

MONODROMY MAP FOR TROPICAL DOLBEAULT COHOMOLOGY

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ABSTRACT. We define monodromy maps for tropical Dolbeault cohomology of algebraic varieties over non-Archimedean fields. We propose a conjecture of Hodge isomorphisms via monodromy maps, and provide some evidence.

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1. INTRODUCTION

Let K be a complete non-Archimedean field. For an algebraic variety X over K , one has the associated K -analytic space X^{an} in the sense of Berkovich [Ber93]. Using a bicomplex $\mathcal{A}_{X^{\text{an}}}^{\bullet,\bullet}$ of \mathbf{R} -sheaves of real forms on X^{an} concentrated in the first quadrant, constructed by Chambert-Loir and Ducros [CLD12], one can define, analogous to complex manifolds, Dolbeault cohomology groups $H_{\text{trop}}^{p,q}(X)$. We call them *tropical Dolbeault cohomology*. They are real vector spaces satisfying $H_{\text{trop}}^{p,q}(X) \neq 0$ only when $0 \leq p, q \leq \dim X$. In this article, we consider a natural question, motivated from complex geometry, about the Hodge isomorphism: Do we have $H_{\text{trop}}^{p,q}(X) \simeq H_{\text{trop}}^{q,p}(X)$?

However, the perspective we take is very different from complex geometry. In fact, we will construct for $p \geq 1$ and $q \geq 0$ a functorial map

$$N_X: H_{\text{trop}}^{p,q}(X) \rightarrow H_{\text{trop}}^{p-1,q+1}(X)$$

which we call the *monodromy map*, via constructing a canonical map $\mathcal{A}_{X^{\text{an}}}^{p,q} \rightarrow \mathcal{A}_{X^{\text{an}}}^{p-1,q+1}$ on the level of sheaves. Such map does not exist in complex geometry. We conjecture (see Conjecture 5.2) that over certain field K , the iterated map

$$N_X^{p-q}: H_{\text{trop}}^{p,q}(X) \rightarrow H_{\text{trop}}^{q,p}(X)$$

is an isomorphism for $p \geq q$ (assuming X is complete and smooth). We view such isomorphism as the Hodge isomorphism for tropical Dolbeault cohomology. If this holds, then we

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have the following numerical relation for corresponding Hodge numbers:

$$h_{\text{trop}}^{p,0}(X) \leq h_{\text{trop}}^{p-1,1}(X) \leq \cdots \geq h_{\text{trop}}^{1,p-1}(X) \geq h_{\text{trop}}^{0,p}(X).$$

This is apparently new from complex geometry. In fact, we should compare the Hodge isomorphism and the above numerical relation with the monodromy-weight conjecture for the E_2 page of the weight spectral sequence (Conjecture 4.4). We remark that the map $N_X^p: H_{\text{trop}}^{p,0}(X) \rightarrow H_{\text{trop}}^{0,p}(X)$ is simply induced by the “flipping” map J [CLD12] multiplied by $p!$ (Lemma 3.2 (1)).

The theorem below provides certain evidence toward our expected Hodge isomorphism. The proof uses arithmetic geometry, especially the weight spectral sequence. The method is restrictive in the sense that any further extension would seem to be relied on the monodromy-weight conjecture in the mixed characteristic case, among other difficulties. Therefore, it is very interesting and important to find a somewhat analytic proof of the results here. However, the situation is rather subtle since we know for some field K , such as the completed algebraic closure of $\mathbf{C}((t))$, the map $N_X: H_{\text{trop}}^{1,0}(X) \rightarrow H_{\text{trop}}^{0,1}(X)$ may fail to be an isomorphism (see Remark 5.3).

In this article, by a non-Archimedean field, we mean a complete topological field equipped with a nontrivial rank-1 non-Archimedean valuation. For a non-Archimedean field K , we denote by K° the ring of integers, \widehat{K} the residue field, K^a a fixed algebraic closure with $\widehat{K^a}$ its completion.

Theorem 1.1. *Let X_0 be a proper smooth scheme over a non-Archimedean field K_0 . Let K be a closed subfield of $\widehat{K_0^a}$ containing K_0 . Put $X = X_0 \otimes_{K_0} K$.*

- (1) (Corollary 5.11) *Suppose that K_0 is isomorphic to $k((t))$ for k either a finite field or a field of characteristic zero. Then the monodromy map*

$$N_X^p: H_{\text{trop}}^{p,0}(X) \rightarrow H_{\text{trop}}^{0,p}(X)$$

is injective for every $p \geq 0$. In particular, $H_{\text{trop}}^{p,0}(X)$ is of finite dimension.

- (2) (Theorem 5.12) *Suppose that K_0 is either a finite extension of \mathbf{Q}_p or isomorphic to $k((t))$ for k a finite field, $K = \widehat{K_0^a}$, and X_0 admits a proper strictly semistable model over K_0° . Then the monodromy map*

$$N_X: H_{\text{trop}}^{1,0}(X) \rightarrow H_{\text{trop}}^{0,1}(X)$$

is an isomorphism.

Remark 1.2. Let K be an algebraically closed non-Archimedean field.

- (1) In his thesis, Jell proved that for a proper smooth scheme X over K of dimension n , the map $N_X^p: H_{\text{trop}}^{p,0}(X) \rightarrow H_{\text{trop}}^{0,p}(X)$ is injective for $p = 0, 1, n$ [Jel16b, Proposition 3.4.11].
- (2) In [JW16], Jell and Wanner proved that for X either \mathbf{P}_K^1 or a (proper smooth) Mumford curve over K , the map $N_X: H_{\text{trop}}^{1,0}(X) \rightarrow H_{\text{trop}}^{0,1}(X)$ is an isomorphism.

Their methods in both papers are analytic.

In [MZ13], Mikhalkin and Zharkov study the tropical homology groups $H_{p,q}(X)$ of a tropical space X . Assuming X compact, they define a map

$$\phi \cap: H_{p,q}(X) \otimes \mathbf{R} \rightarrow H_{p+1,q-1}(X) \otimes \mathbf{R}$$

using their eigenwave ϕ . In fact, as shown in [JSS15], one can compute the tropical homology $H_{p,q}(X) \otimes \mathbf{R}$, or rather the tropical cohomology, via superforms in [CLD12] as well. Then our construction in §2 would give rise to a map $N_X: H_{p,q}(X) \otimes \mathbf{R} \rightarrow H_{p+1,q-1}(X) \otimes \mathbf{R}$ (or rather on cohomology) for any tropical space X . We expect that N_X and $\phi \cap$ should coincide when X is compact, possibly up to an elementary factor.

Moreover, in [MZ13], the authors prove that when X is a realizable smooth compact tropical space, the iterated map $\phi^{q-p} \cap: H_{p,q}(X) \otimes \mathbf{R} \rightarrow H_{q,p}(X) \otimes \mathbf{R}$ is an isomorphism. They first realize X as the tropical limit of a complex projective one-parameter semistable degeneration \mathcal{X} such that all strata of the singular fiber \mathcal{X}_0 are blow-ups of projective spaces. By the work [IKMZ16], one knows that $H_{p,q}(X) \otimes \mathbf{Q}$ can be identified with E_2 -terms of the Steenbrink–Illusie spectral sequence associated to \mathcal{X} . Moreover, Mikhalkin and Zharkov show that under such identification, the map $\phi \cap$ is simply the monodromy map on E_2 -terms. In our work, X is a proper smooth¹ algebraic variety and we relate our tropical Dolbeault cohomology $H_{\text{trop}}^{p,q}(X)$ to E_2 -terms of the weight spectral sequence of (various) semistable alteration \mathcal{X} of X (see Definition 5.4 for the precise meaning) for certain p, q , under which N_X is essentially the monodromy map. In our setup, the semistable scheme \mathcal{X} is very general, so that its E_2 -terms cannot be read off solely from the dual complex of \mathcal{X} , hence we do not obtain a strict identification for general p, q . Nevertheless, it would be interesting to compare our approach relating to the weight spectral sequence and their approach relating to the Steenbrink–Illusie spectral sequence in [MZ13, IKMZ16].

Convention. For an analytic space X over K , we use $H^\bullet(X, -)$ to indicate the cohomology group with respect to the underlying topology of X , and $H_{\text{ét}}^\bullet(X, -)$ to indicate the cohomology group with respect to the étale topology of X .

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2. MONODROMY MAPS FOR SUPERFORMS ON VECTOR SPACES

In this section, we review the construction of superforms on vector spaces from [CLD12] and introduce the corresponding monodromy map.

Let V be an \mathbf{R} -vector space of dimension n . Let T_V be the tangent space of V and T_V^* its dual space. For every open subset U of V and integers $p, q \geq 0$, we have the space of (p, q) -forms on U ,

$$\mathcal{A}_V^{p,q}(U) := \mathcal{A}_V(U) \otimes_{\mathbf{R}} \wedge^p T_V^* \otimes_{\mathbf{R}} \wedge^q T_V^*,$$

where $\mathcal{A}_V(U)$ is the \mathbf{R} -algebra of smooth functions on U . The direct sum $\mathcal{A}_V^{*,*}(U) := \bigoplus \mathcal{A}_V^{p,q}(U)$ form a bigraded \mathbf{R} -algebra with the following commutativity law: if ω is a (p, q) -form and ω' is a (p', q') -form, then

$$\omega' \wedge \omega = (-1)^{(p+q)(p'+q')} \omega \wedge \omega'.$$

Moreover, $\wedge: \mathcal{A}_V^{p,0}(U) \times \mathcal{A}_V^{0,q}(U) \rightarrow \mathcal{A}_V^{p,q}(U)$ is simply the one induced by the tensor product map.

¹It is worth noting that the tropicalization of a smooth analytic space, such as X^{an} , under a local tropical chart is in general not a smooth tropical space.

The real vector spaces $\mathcal{A}_V^{\bullet,\bullet}(U)$ form a bicomplex with two differentials

$$d': \mathcal{A}_V^{p,q}(U) \rightarrow \mathcal{A}_V^{p+1,q}(U), \quad d'': \mathcal{A}_V^{p,q}(U) \rightarrow \mathcal{A}_V^{p,q+1}(U).$$

We also have a convolution map $J: \mathcal{A}_V^{p,q}(U) \rightarrow \mathcal{A}_V^{q,p}(U)$.

In terms of coordinates, they are described as follows. Let $\{x_1, \dots, x_n\}$ be a system of coordinates of V . Then a (p, q) -form can be written as

$$\omega = \sum_{|I|=p, |J|=q} \omega_{I,J}(x) d'x_I \otimes d''x_J = \sum_{|I|=p, |J|=q} \omega_{I,J}(x) d'x_I \wedge d''x_J$$

where I, J are subsets of $\{1, \dots, n\}$, and $\omega_{I,J}(x)$ are smooth functions on U . For such ω , we have

$$\begin{aligned} d'\omega &= \sum_{|I|=p, |J|=q} \sum_{i=1}^n \frac{\partial \omega_{I,J}(x)}{\partial x_i} d'x_i \wedge d'x_I \wedge d''x_J, \\ d''\omega &= (-1)^p \sum_{|I|=p, |J|=q} \sum_{j=1}^n \frac{\partial \omega_{I,J}(x)}{\partial x_j} d'x_I \wedge d''x_j \wedge d''x_J, \\ J\omega &= (-1)^{pq} \sum_{|I|=p, |J|=q} \omega_{I,J}(x) d'x_J \wedge d''x_I. \end{aligned}$$

We remind readers from [CLD12, §1.2] the relations

$$d'' = Jd'J, \quad Jd' = d''J, \quad Jd'' = d'J, \quad d' = Jd''J,$$

and for a (p, q) -form ω and a (p', q') -form ω' ,

$$\begin{aligned} d'(\omega \wedge \omega') &= d'\omega \wedge \omega' + (-1)^{p+q} \omega \wedge d'\omega', \\ d''(\omega \wedge \omega') &= d''\omega \wedge \omega' + (-1)^{p+q} \omega \wedge d''\omega'. \end{aligned}$$

Now we are going to define a map $N: \mathcal{A}_V^{p,q}(U) \rightarrow \mathcal{A}_V^{p-1,q+1}(U)$ for $p \geq 1$. We first recall the notion of coevaluation map. Let W be an arbitrary finite dimensional \mathbf{R} -vector space with W^* its dual space. We have a canonical evaluation map

$$\text{ev}: W^* \otimes_{\mathbf{R}} W \rightarrow \mathbf{R}.$$

We define the *coevaluation map* to be the unique linear map

$$\text{coev}: \mathbf{R} \rightarrow W \otimes_{\mathbf{R}} W^*$$

such that both composite maps

$$\begin{aligned} W^* &\xrightarrow{1_{W^*} \otimes \text{coev}} W^* \otimes_{\mathbf{R}} (W \otimes_{\mathbf{R}} W^*) \xrightarrow{\sim} (W^* \otimes_{\mathbf{R}} W) \otimes_{\mathbf{R}} W^* \xrightarrow{\text{ev} \otimes 1_{W^*}} W^* \\ W &\xrightarrow{\text{coev} \otimes 1_W} (W \otimes_{\mathbf{R}} W^*) \otimes_{\mathbf{R}} W \xrightarrow{\sim} W \otimes_{\mathbf{R}} (W^* \otimes_{\mathbf{R}} W) \xrightarrow{1_W \otimes \text{ev}} W \end{aligned}$$

are identity maps. Now we apply the coevaluation map to the vector space $W = T_V$.

Definition 2.1. Let $p \geq 1$ be an integer. Define the map

$$N: \mathcal{A}_V^{p,q}(U) \rightarrow \mathcal{A}_V^{p-1,q+1}(U)$$

to be the composite map

$$\begin{aligned}
\mathcal{C}^\infty(U) \otimes_{\mathbf{R}} \wedge^p T_V^* \otimes_{\mathbf{R}} \wedge^q T_V^* &\xrightarrow{\sim} \mathcal{C}^\infty(U) \otimes_{\mathbf{R}} \wedge^p T_V^* \otimes_{\mathbf{R}} \mathbf{R} \otimes_{\mathbf{R}} \wedge^q T_V^* \\
&\rightarrow \mathcal{C}^\infty(U) \otimes_{\mathbf{R}} \wedge^p T_V^* \otimes_{\mathbf{R}} (T_V \otimes_{\mathbf{R}} T_V^*) \otimes_{\mathbf{R}} \wedge^q T_V^* \\
&\xrightarrow{\sim} \mathcal{C}^\infty(U) \otimes_{\mathbf{R}} (\wedge^p T_V^* \otimes_{\mathbf{R}} T_V) \otimes_{\mathbf{R}} (T_V^* \otimes_{\mathbf{R}} \wedge^q T_V^*) \\
&\rightarrow \mathcal{C}^\infty(U) \otimes_{\mathbf{R}} \wedge^{p-1} T_V^* \otimes_{\mathbf{R}} \wedge^{q+1} T_V^*,
\end{aligned}$$

where the second map is given by the coevaluation map for T_V , and the last map is given by the contraction map and the wedge product.

For $0 \leq r \leq p$, we denote by $N^r: \mathcal{A}_V^{p,q}(U) \rightarrow \mathcal{A}_V^{p-r,q+r}(U)$ the consecutive composition.

In terms of the coordinates $\{x_1, \dots, x_n\}$ of V , we have for

$$(2.1) \quad \omega = \sum_{I=\{i_1 < \dots < i_p\}, J=\{j_1 < \dots < j_q\}} \omega_{I,J}(x) d'x_{i_1} \wedge \dots \wedge d'x_{i_p} \wedge d''x_{j_1} \wedge \dots \wedge d''x_{j_q}$$

with $p \geq 1$ that

$$\begin{aligned}
N\omega &= \sum_{k=1}^p \sum_{I,J} (-1)^{p-k} \omega_{I,J}(x) d'x_{i_1} \wedge \dots \wedge \widehat{d'x_{i_k}} \wedge \dots \wedge d'x_{i_p} \wedge d''x_{i_k} \wedge d''x_{j_1} \wedge \dots \wedge d''x_{j_q} \\
&= \sum_{k=1}^p \sum_{I,J} (-1)^{p-k} \omega_{I,J}(x) d'x_{I \setminus \{i_k\}} \wedge d''x_{i_k} \wedge d''x_J.
\end{aligned}$$

Lemma 2.2. *We have $Nd'' = d''N: \mathcal{A}_V^{p,q}(U) \rightarrow \mathcal{A}_V^{p-1,q+2}(U)$ for $p \geq 1$.*

Proof. Take a (p, q) -form ω as (2.1). We have

$$\begin{aligned}
d''N\omega &= d'' \sum_{k=1}^p \sum_{I,J} (-1)^{p-k} \omega_{I,J}(x) d'x_{I \setminus \{i_k\}} \wedge d''x_{i_k} \wedge d''x_J \\
&= - \sum_{I,J} \sum_{j=1}^n \sum_{k=1}^p \frac{(-1)^k \partial \omega_{I,J}(x)}{\partial x_j} d'x_{I \setminus \{i_k\}} \wedge d''x_j \wedge d''x_{i_k} \wedge d''x_J.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
Nd''\omega &= N(-1)^p \sum_{I,J} \sum_{j=1}^n \frac{\partial \omega_{I,J}(x)}{\partial x_j} d'x_I \wedge d''x_j \wedge d''x_J \\
&= \sum_{k=1}^p \sum_{I,J} \sum_{j=1}^n \frac{(-1)^k \partial \omega_{I,J}(x)}{\partial x_j} d'x_{I \setminus \{i_k\}} \wedge d''x_{i_k} \wedge d''x_j \wedge d''x_J \\
&= - \sum_{I,J} \sum_{j=1}^n \sum_{k=1}^p \frac{(-1)^k \partial \omega_{I,J}(x)}{\partial x_j} d'x_{I \setminus \{i_k\}} \wedge d''x_j \wedge d''x_{i_k} \wedge d''x_J.
\end{aligned}$$

The lemma follows. \square

Lemma 2.3. *For a (p, q) -form ω and a (p', q') -form ω' with $p, p' \geq 1$ and $p + p' \geq n + 1$, we have*

$$N\omega \wedge \omega' = -\omega \wedge N\omega'.$$

Proof. By linearity, we may assume that

$$\omega = \omega(x) d'x_I \wedge d''x_J, \quad \omega' = \omega'(x) d'x_{I'} \wedge d''x_{J'}.$$

If $|I \cap I'| \geq 2$, then it is easy to see that $N\omega \wedge \omega' = \omega \wedge N\omega' = 0$. Otherwise, $|I \cap I'| = 1$. Without loss of generality, we may assume that $I \cap I' = \{n\}$ hence $d'x_I = d'x_{I \setminus \{n\}} \wedge d'x_n$ and $d'x_{I'} = d'x_{I' \setminus \{n\}} \wedge d'x_n$. Then we have

$$\begin{aligned} N\omega \wedge \omega' &= (\omega(x)d'x_{I \setminus \{n\}} \wedge d''x_n \wedge d''x_J) \wedge (\omega'(x)d'x_{I'} \wedge d''x_{J'}) \\ &= (-1)^{p'(q+1)}\omega(x)\omega'(x)d'x_{I \setminus \{n\}} \wedge d'x_{I'} \wedge d''x_n \wedge d''x_J \wedge d''x_{J'} \\ &= (-1)^{p'(q+1)}\omega(x)\omega'(x)d'x_{I \setminus \{n\}} \wedge d'x_{I' \setminus \{n\}} \wedge d'x_n \wedge d''x_n \wedge d''x_J \wedge d''x_{J'}; \end{aligned}$$

and

$$\begin{aligned} \omega \wedge N\omega' &= (\omega(x)d'x_I \wedge d''x_J) \wedge (\omega'(x)d'x_{I' \setminus \{n\}} \wedge d''x_n \wedge d''x_{J'}) \\ &= (-1)^{p'q}\omega(x)\omega'(x)d'x_I \wedge d'x_{I' \setminus \{n\}} \wedge d''x_n \wedge d''x_J \wedge d''x_{J'} \\ &= (-1)^{p'q}\omega(x)\omega'(x)d'x_{I \setminus \{n\}} \wedge d'x_n \wedge d'x_{I' \setminus \{n\}} \wedge d''x_n \wedge d''x_J \wedge d''x_{J'} \\ &= (-1)^{p'q+(p'-1)}\omega(x)\omega'(x)d'x_{I \setminus \{n\}} \wedge d'x_{I' \setminus \{n\}} \wedge d'x_n \wedge d''x_n \wedge d''x_J \wedge d''x_{J'}. \end{aligned}$$

The lemma follows. \square

Lemma 2.4. *Let V' be another \mathbf{R} -vector space of dimension n' and $U' \subset V'$ an subset. Let $\varphi: V' \rightarrow V$ be an affine map such that $\varphi(U') \subset U$. Then for $p \geq 1$, we have*

$$N'\varphi^* = \varphi^*N: \mathcal{A}_V^{p,q}(U) \rightarrow \mathcal{A}_{V'}^{p-1,q+1}(U')$$

where N' denotes the monodromy map for V' .

Proof. We may assume that φ is a linear map. It suffices to consider cases where φ is injective or surjective.

Suppose that φ is injective. Regard V' as a subspace of V . Choose a basis $\{x_1, \dots, x_n\}$ of V such that V' is spanned by $\{x_1, \dots, x_{n'}\}$ (with $n' \leq n$). Take $\omega = \omega(x)d'x_I \wedge d''x_J \in \mathcal{A}_V^{p,q}(U)$ for $I = \{i_1 < \dots < i_p\}$. If $i_p > n'$, then $N'\varphi^*\omega = \varphi^*N\omega = 0$. If $i_p \leq n'$, then we have

$$N'\varphi^*\omega = \varphi^*N\omega = \left(\omega(x) \sum_{k=1}^p (-1)^{p-k} d'x_{I \setminus \{i_k\}} \wedge d''x_{i_k} \wedge d''x_J \right) |_{U'}.$$

Suppose that φ is surjective. Choose a basis $\{x_1, \dots, x_{n'}\}$ of V' such that $\ker \varphi$ is spanned by $\{x_{n'+1}, \dots, x_n\}$ (with $n \leq n'$). We identify V with the subspace of V' spanned by $\{x_1, \dots, x_n\}$. Then again we have

$$N'\varphi^*\omega = \varphi^*N\omega = \left(\omega(\varphi(x)) \sum_{k=1}^p (-1)^{p-k} d'x_{I \setminus \{i_k\}} \wedge d''x_{i_k} \wedge d''x_J \right) |_{U'}.$$

The lemma follows. \square

Lemma 2.5. *The map $N^p: \mathcal{A}_V^{p,0}(U) \rightarrow \mathcal{A}_V^{0,p}(U)$ coincides with $p! \cdot J: \mathcal{A}_V^{p,0}(U) \rightarrow \mathcal{A}_V^{0,p}(U)$.*

Proof. It is elementary. \square

Remark 2.6. The presheaf $U \mapsto \mathcal{A}_V^{p,q}(U)$ is already a sheaf on V . The maps d', d'', J, \wedge, N induce maps of sheaves with same relations. In particular, we have the map

$$N_V: \mathcal{A}_V^{p,q} \rightarrow \mathcal{A}_V^{p-1,q+1}$$

of sheaves for $p \geq 1$, and corresponding Lemma 2.2 and Lemma 2.3. Lemma 2.4 induces the equality

$$(\varphi_* N_{V'})\varphi^* = \varphi^* N_V: \mathcal{A}_V^{p,q} \rightarrow \varphi_* \mathcal{A}_{V'}^{p-1,q+1}$$

on the level of sheaves.

Let P be a polyhedral complex (*polytope* in [CLD12, §1.1]) in V and $j: P \rightarrow V$ the tautological inclusion map. For every open subset U of P , let $\mathcal{N}^{p,q}(U)$ be the subspace of $(j^{-1}\mathcal{A}_V^{p,q})(U)$ of (p, q) -forms ω such that for every polyhedron C of P , the restriction of ω to $\langle C \rangle$ is zero on $\langle C \rangle \cap U$ where $\langle C \rangle$ is the affine subset of V spanned by C . It is clear that $\mathcal{N}^{p,q}$ is a \mathbf{R} -subsheaf of $j^{-1}\mathcal{A}_V^{p,q}$. Let $\mathcal{A}_P^{p,q}$ be the quotient sheaf $j^{-1}\mathcal{A}_V^{p,q}/\mathcal{N}^{p,q}$.

By Lemma 2.4, the monodromy map $j^{-1}N_V: j^{-1}\mathcal{A}_V^{p,q} \rightarrow j^{-1}\mathcal{A}_V^{p,q}$ preserves $\mathcal{N}^{p,q}$. Therefore, we have an induced map

$$N_P: \mathcal{A}_P^{p,q} \rightarrow \mathcal{A}_P^{p-1,q+1}$$

of sheaves for $p \geq 1$.

3. MONODROMY MAPS FOR REAL FORMS ON ANALYTIC SPACES

In this section, we review the construction of (p, q) -forms in [CLD12], tropical Dolbeault cohomology, and introduce the monodromy map on analytic spaces.

We fix a non-Archimedean field K . Let X be a K -analytic space. Recall that a *tropical chart* of X is given by a moment map $f: X \rightarrow T$ to a torus T over K and a compact polyhedral complex P of T_{trop} that contains $f_{\text{trop}}(X)$. Here T_{trop} is the tropicalization of T , which is a \mathbf{R} -vector space of finite dimension, and $f_{\text{trop}}: X \rightarrow T \rightarrow T_{\text{trop}}$ is the composite map.

For every open subset U of X , denote by $\mathcal{A}_{\text{pre}}^{p,q}(U)$ the inductive limit of $\mathcal{A}_P^{p,q}(P)$ for all tropical charts $(f: U \rightarrow T, P)$ of U . Again by Lemma 2.4, the monodromy maps N_P are compatible with transition maps. Therefore, we obtain a map $N_{\text{pre}}: \mathcal{A}_{\text{pre}}^{p,q}(U) \rightarrow \mathcal{A}_{\text{pre}}^{p-1,q+1}(U)$ for $p \geq 1$. The sheaf of (p, q) -forms on X is defined as the sheafification of $U \mapsto \mathcal{A}_{\text{pre}}^{p,q}(U)$, denoted by $\mathcal{A}_X^{p,q}$.

Definition 3.1. For $p \geq 1$, we define the *monodromy map* for forms on X , denoted by

$$N_X: \mathcal{A}_X^{p,q} \rightarrow \mathcal{A}_X^{p-1,q+1}$$

to be the sheafification of $U \mapsto [N_{\text{pre}}: \mathcal{A}_{\text{pre}}^{p,q}(U) \rightarrow \mathcal{A}_{\text{pre}}^{p-1,q+1}(U)]$.

For $0 \leq r \leq p$, we denote by $N_X^r: \mathcal{A}_X^{p,q} \rightarrow \mathcal{A}_X^{p-r,q+r}$ the iterated composition.

Lemma 3.2. *We have*

- (1) The map $N_X^p: \mathcal{A}_X^{p,q} \rightarrow \mathcal{A}_X^{0,p}$ coincides with $p! \cdot J$.
- (2) N_X commutes with d'' , that is, $N_X d'' = d'' N_X: \mathcal{A}_X^{p,q} \rightarrow \mathcal{A}_X^{p-1,q+2}$ for $p \geq 1$.
- (3) $(N_X -) \wedge - = -(- \wedge (N_X -))$: $\mathcal{A}_X^{p,q} \times \mathcal{A}_X^{p',q'} \rightarrow \mathcal{A}_X^{p+p'-1,q+q'+1}$ for $p, p' \geq 1$ and $p + p' \geq \dim X + 1$.
- (4) Let $\varphi: X' \rightarrow X$ be a map of K -analytic spaces. Then

$$(\varphi_* N_{X'}) \varphi^* = \varphi^* N_X: \mathcal{A}_X^{p,q} \rightarrow \varphi_* \mathcal{A}_{X'}^{p-1,q+1}.$$

Proof. They are consequences of Lemma 2.5, Lemma 2.2, Lemma 2.3 and Lemma 2.4, respectively. \square

For a fixed integer p , we have the complex

$$(\mathcal{A}_X^{p,\bullet}, d''): \mathcal{A}_X^{p,0} \xrightarrow{d''} \mathcal{A}_X^{p,1} \xrightarrow{d''} \dots$$

Definition 3.3 (Dolbeault cohomology, [Liu17]). Let X be a K -analytic space. We define the *Dolbeault cohomology* (of forms) to be

$$H^{p,q}(X) := \frac{\ker(d'' : \mathcal{A}_X^{p,q}(X) \rightarrow \mathcal{A}_X^{p+1}(X))}{\operatorname{im}(d'' : \mathcal{A}_X^{p,q-1}(X) \rightarrow \mathcal{A}_X^{p,q}(X))}.$$

The monodromy map in Definition 3.1 is in fact a map of complexes

$$N_X : \mathcal{A}_X^{p,\bullet} \rightarrow \mathcal{A}_X^{p-1,\bullet}[1],$$

and thus induces a map

$$N_X : H^{p,q}(X) \rightarrow H^{p-1,q+1}(X)$$

of Dolbeault cohomology when $p \geq 1$.

Let \mathcal{O}_X be the structure sheaf of X . For $p \geq 0$, let $\mathcal{O}_X^{(p)}$ be the sheaf such that for every open subset U of X , $\mathcal{O}_X^{(p)}(U)$ is the \mathbf{Q} -vector space spanned by symbols $\{f_1, \dots, f_p\}$ with $f_i \in \mathcal{O}_X^*(U)$. For $p \geq 0$, we have a natural map

$$\tau : \mathcal{O}_X^{(p)} \rightarrow \ker[d'' : \mathcal{A}_X^{p,0} \rightarrow \mathcal{A}_X^{p,1}]$$

of \mathbf{Q} -sheaves on X . Let \mathcal{T}_X^p be its image sheaf. We recall the definition of τ . For an open subset U of X and $f_1, \dots, f_p \in \mathcal{O}_X^*(U)$, we have a moment map $f = (f_1, \dots, f_p) : U \rightarrow T = (\mathbf{G}_m^{\text{an}})^p$. Let $\{x_1, \dots, x_p\}$ be the standard coordinates of $T_{\text{trop}} = \mathbf{R}^p$ ². Then $\tau(\{f_1, \dots, f_p\})$ is defined as $d'x_1 \wedge \dots \wedge d'x_p$, regarded as an element in $\ker[d'' : \mathcal{A}_X^{p,0}(U) \rightarrow \mathcal{A}_X^{p,1}(U)]$.

Proposition 3.4. *The canonical map*

$$\mathcal{T}_X^p \otimes_{\mathbf{Q}} \mathbf{R} \rightarrow \ker[d'' : \mathcal{A}_X^{p,0} \rightarrow \mathcal{A}_X^{p,1}]$$

is an isomorphism. It induces a canonical isomorphism

$$H^q(X, \mathcal{T}_X^p) \otimes_{\mathbf{Q}} \mathbf{R} \simeq H^{p,q}(X).$$

Proof. By [Jel16a, Corollary 4.6] and [CLD12, Corollaire 3.3.7], the complex $(\mathcal{A}_X^{p,\bullet}, d'')$ is a fine resolution of $\ker[d'' : \mathcal{A}_X^{p,0} \rightarrow \mathcal{A}_X^{p,1}]$. Thus the proposition follows from standard facts in homological algebra. \square

4. SEMISTABLE SCHEMES AND COHOMOLOGICAL MONODROMY MAPS

In this section, we introduce some constructions for strictly semistable schemes, and review the (cohomological) monodromy maps coming from the weight spectral sequence.

Let K be a discrete non-Archimedean field, that is, a non-Archimedean field with discrete valuation. Fix a rational prime ℓ that is invertible in \widetilde{K} . Denote by $K^{\text{ur}} \subset K^{\text{a}}$ the maximal unramified extension with the residue field \widetilde{K}^s , which is a separable closure of \widetilde{K} . Let $I_K \subset \operatorname{Gal}(K^{\text{a}}/K)$ be the inertia subgroup, so the quotient group $\operatorname{Gal}(K^{\text{a}}/K)/I_K$ can be identified with $\operatorname{Gal}(\widetilde{K}^s/\widetilde{K})$. Denote by $t_\ell : I_K \rightarrow \mathbf{Z}_\ell(1)$ the $(\ell$ -adic) tame quotient homomorphism, that is, the one sending $\sigma \in I_K$ to $(\sigma(\varpi^{1/\ell^n})/\varpi^{1/\ell^n})_n$ for a uniformizer ϖ of K . We fix an element $T \in I_K$ such that $t_\ell(T)$ is a topological generator of $\mathbf{Z}_\ell(1)$.

For a separated scheme X of finite type over K , put $X_{\text{a}} = X \otimes_K K^{\text{a}}$, and let X^{an} be the associated K -analytic space ([Ber93]). Put $X_{\text{a}}^{\text{an}} = (X \otimes_K \widetilde{K}^{\text{a}})^{\text{an}}$. For a scheme \mathcal{X} over K° , put $\mathcal{X}_\eta = \mathcal{X} \otimes_{K^\circ} K$. For a scheme Z over \widetilde{K} , put $Z_s = Z \otimes_{\widetilde{K}} \widetilde{K}^s$.

We first recall the following definition (see [Sai03] for example).

²The map $\mathbf{G}_m^{\text{an}} \rightarrow (\mathbf{G}_m^{\text{an}})_{\text{trop}} = \mathbf{R}$ is given by $-\log || \cdot ||$.

Definition 4.1 (Strictly semistable scheme). Let \mathcal{X} be a scheme locally of finite presentation over $\text{Spec } K^\circ$. We say that \mathcal{X} is *strictly semistable* if it is Zariski locally smooth over

$$\text{Spec } K^\circ[t_0, \dots, t_p]/(t_0 \cdots t_p - \varpi)$$

for some integer $p \geq 0$ (which may vary) and a uniformizer ϖ of K .

Let \mathcal{X} be a proper strictly semistable scheme over K° . The special fiber $Y := \mathcal{X} \otimes_{K^\circ} \widetilde{K}$ is a normal crossing divisor of \mathcal{X} . Suppose that $\{Y^1, \dots, Y^m\}$ is the set of irreducible components of Y . For a nonempty subset $I \subset \{1, \dots, m\}$, put $Y^I = \bigcap_{i \in I} Y^i$. For $p \geq 0$, put

$$Y^{(p)} = \coprod_{I \subset \{1, \dots, m\}, |I|=p+1} Y^I.$$

Then $Y^{(p)}$ is a finite disjoint union of smooth proper subschemes of Y of codimension p over \widetilde{K} . Denote by $Y^{[p]}$ the image of the canonical morphism $Y^{(p)} \rightarrow Y$.

Notation 4.2. For a subscheme Z of Y , put $I(Z) = \{i \in \{1, \dots, m\} \mid Z \subset Y^i\}$ and $p(Z) = |I(Z)| - 1$.

Construction 4.3. Let y be a point of Y with $p = p(y) \geq 0$. Suppose that $I(y) = \{i_0 < \dots < i_p\}$ (Notation 4.2). Choose an affine open neighborhood \mathcal{V} of y in \mathcal{X} such that \mathcal{V} is smooth over $\text{Spec } K^\circ[t_0, \dots, t_p]/(t_0 \cdots t_p - \varpi)$ under which the divisor defined by t_j is $\mathcal{V} \cap Y^{i_j}$. Let g_j be the restriction of t_j to \mathcal{V}_η .

We say that $(\mathcal{V}, \{g_0, \dots, g_p\})$ is a *semistable chart* at y if moreover $\mathcal{V} \cap W$ is either empty or connected for every irreducible component W of $Y^{[r]}$ with $r \geq 0$.

For subsets $J \subset I \subset \{1, \dots, m\}$ such that $|I| = |J| + 1$, let $i_{JI}: \bigcap_{i \in I} Y^i \rightarrow \bigcap_{i \in J} Y^i$ denote the closed immersion. If $I = \{i_0 < \dots < i_p\}$ and $J = I \setminus \{i_j\}$, then we put $\epsilon(J, I) = (-1)^j$. We define the *pullback map*

$$(4.1) \quad \delta_p^*: H_{\text{ét}}^q(Y_s^{(p)}, \mathbf{Q}_\ell) \rightarrow H_{\text{ét}}^q(Y_s^{(p+1)}, \mathbf{Q}_\ell)$$

to be the alternating sum $\sum_{I \subset J, |I|=|J|-1=p+1} \epsilon(I, J) i_{IJ}^*$ of the restriction maps, and the *push-forward map*

$$(4.2) \quad \delta_{p*}: H_{\text{ét}}^q(Y_s^{(p)}, \mathbf{Q}_\ell) \rightarrow H_{\text{ét}}^{q+2}(Y_s^{(p-1)}, \mathbf{Q}_\ell(1))$$

to be the alternating sum $\sum_{I \supset J, |I|=|J|+1=p+1} \epsilon(J, I) i_{JI*}$ of the Gysin maps. These maps satisfy the formula

$$\delta_{p-1}^* \circ \delta_{p*} + \delta_{p+1*} \circ \delta_p^* = 0.$$

Let us recall the following weight spectral sequence $E_{\mathcal{X}}^{p,q}$ attached to \mathcal{X}^3 , originally studied in [RZ82]:

$$E_{\mathcal{X},1}^{p,q} = \bigoplus_{i \geq \max(0, -p)} H_{\text{ét}}^{q-2i}(Y_s^{(p+2i)}, \mathbf{Q}_\ell(-i)) \Rightarrow H_{\text{ét}}^{p+q}(\mathcal{X}_{\eta, \text{a}}, \mathbf{Q}_\ell).$$

Here we will follow the convention and discussion in [Sai03]. When \mathcal{X} is fixed, we write E for $E_{\mathcal{X}}$ for short. By [Sai03, Corollary 2.8 (2)], we have a map $\mu: E_{\bullet}^{-1, \bullet+1} \rightarrow E_{\bullet}^{\bullet+1, \bullet-1}$ of spectral sequences (depending on T). The differential map $d_1^{p,q}$ is an appropriate sum of pullback and pushforward maps. The map $\mu_1^{p,q}: E_1^{p-1, q+1} \rightarrow E_1^{p+1, q-1}$ is the sum of its restrictions to each direct summand $H_{\text{ét}}^{q+1-2i}(Y_s^{(2i+1)}, \mathbf{Q}_\ell(-i))$, and such restriction is the tensor product map by

³It also depends on the ordering of the set of irreducible components of Y .

$t_\ell(T)$ (resp. the zero map) if $H_{\text{ét}}^{q+1-2i}(Y_s^{(2i+1)}, \mathbf{Q}_\ell(-i+1))$ does (resp. does not) appear in the target. For integers p, q and $r \geq 0$, the map μ induces a map

$$N_{\mathcal{X}}^r: E_2^{p,q}(r) \rightarrow E_2^{p+2r,q-2r}$$

which depends only on \mathcal{X} .

Conjecture 4.4 (Weight-monodromy conjecture). *Let \mathcal{X} be a proper strictly semistable scheme over K° . Then for all integers $p, r \geq 0$, the map*

$$N_{\mathcal{X}}^r: E_2^{-r,p+r}(r) \rightarrow E_2^{r,p-r}$$

is an isomorphism.

For $p \geq 1$, the map $N_{\mathcal{X}}^p: E_2^{-p,2p}(p) \rightarrow E_2^{p,0}$ is simply the map

$$(4.3) \quad E_2^{-p,2p}(p) = \ker[H_{\text{ét}}^0(Y_s^{(p)}, \mathbf{Q}_\ell) \xrightarrow{(\delta_p^*, \delta_{p*}^*)} H_{\text{ét}}^0(Y_s^{(p+1)}, \mathbf{Q}_\ell) \oplus H_{\text{ét}}^2(Y_s^{(p-1)}, \mathbf{Q}_\ell(1))] \\ \rightarrow \frac{\ker[H_{\text{ét}}^0(Y_s^{(p)}, \mathbf{Q}_\ell) \xrightarrow{\delta_p^*} H_{\text{ét}}^0(Y_s^{(p+1)}, \mathbf{Q}_\ell)]}{\text{im}[H_{\text{ét}}^0(Y_s^{(p-1)}, \mathbf{Q}_\ell) \xrightarrow{\delta_{p-1}^*} H_{\text{ét}}^0(Y_s^{(p)}, \mathbf{Q}_\ell)]} = E_2^{p,0}$$

induced by the identity map on $H_{\text{ét}}^0(Y_s^{(p)}, \mathbf{Q}_\ell)$.

Proposition 4.5. *We have*

- (1) *The spectral sequence $E_{\mathcal{X}}$ degenerates from the second page. In particular, $E_2^{p,0}$ is canonically a subspace of $H_{\text{ét}}^p(\mathcal{X}_{\eta,a}, \mathbf{Q}_\ell)$.*
- (2) *If K has equal characteristic, then Conjecture 4.4 holds.*
- (3) *If \widetilde{K} is a purely inseparable extension of a finitely generated extension of a prime field, then $N_{\mathcal{X}}: E_2^{-1,2}(1) \rightarrow E_2^{1,0}$ is an isomorphism.*

Proof. See [Ito05, Theorem 1.1] for (1) and (2). Part (3) follows from [Ito05, Proposition 2.5 & Remark 2.4]. \square

Let X be a separated scheme of finite type over K . From [Ber00, §1], we have the following map

$$(4.4) \quad \kappa_X^p: H^p(X^{\text{an}}, \mathbf{Q}_\ell) \rightarrow H_{\text{ét}}^p(X_a, \mathbf{Q}_\ell).$$

It is defined as the composition of the restriction map $H^p(X^{\text{an}}, \mathbf{Q}_\ell) \rightarrow H_{\text{ét}}^p(X_a^{\text{an}}, \mathbf{Q}_\ell)$ and the inverse of the comparison isomorphism $H_{\text{ét}}^p(X_a, \mathbf{Q}_\ell) \xrightarrow{\sim} H_{\text{ét}}^p(X_a^{\text{an}}, \mathbf{Q}_\ell)$.

Lemma 4.6. *The map $\kappa_{\mathcal{X}_\eta}^p$ (4.4) is injective with image contained in $E_2^{p,0}$. Moreover, if every irreducible component of $Y^{(r)}$ ($r \geq 0$) is geometrically irreducible, then $\kappa_{\mathcal{X}_\eta}^p$ induces an isomorphism $H^p(\mathcal{X}_\eta^{\text{an}}, \mathbf{Q}_\ell) \xrightarrow{\sim} E_2^{p,0}$.*

Proof. From [Ber00, §4], we have the following diagram

$$\begin{array}{ccc} H_{\text{Zar}}^p(Y, \mathbf{Q}_\ell) & \xrightarrow{\sim} & H^p(\mathcal{X}_\eta^{\text{an}}, \mathbf{Q}_\ell) \\ \downarrow & & \downarrow \\ H_{\text{Zar}}^p(Y_s, \mathbf{Q}_\ell) & \xrightarrow{\sim} & H^p(\mathcal{X}_{\eta,a}^{\text{an}}, \mathbf{Q}_\ell) \\ \downarrow & & \downarrow \\ H_{\text{ét}}^p(Y_s, \mathbf{Q}_\ell) & \longrightarrow & H_{\text{ét}}^p(\mathcal{X}_{\eta,a}, \mathbf{Q}_\ell) \end{array}$$

in which the composition of the two right vertical maps is just $\kappa_{\mathcal{X}_\eta}^p$. By [Ber00, Lemma 4.1], the upper and middle horizontal maps are both isomorphisms. The map $H^p(\mathcal{X}_\eta^{\text{an}}, \mathbf{Q}_\ell) \rightarrow H^p(\mathcal{X}_{\eta,a}^{\text{an}}, \mathbf{Q}_\ell)$ is injective by the discussion after [Ber00, Theorem 1.1]. By the proper descent and the fact that $H_{\text{Zar}}^i(Z, \mathbf{Q}_\ell) = 0$ for $i > 0$ and Z smooth over \widetilde{K}^s , the composite map

$$H_{\text{Zar}}^p(Y_s, \mathbf{Q}_\ell) \rightarrow H_{\text{ét}}^p(Y_s, \mathbf{Q}_\ell) \rightarrow H_{\text{ét}}^p(\mathcal{X}_{\eta,a}, \mathbf{Q}_\ell)$$

is an isomorphism onto its image, which is $E_2^{p,0}$. Therefore, $\kappa_{\mathcal{X}_\eta}^p$ (4.4) is injective with image contained in $E_2^{p,0}$.

Moreover, if every irreducible component of $Y^{(r)}$ is geometrically irreducible, then the map $H_{\text{Zar}}^p(Y, \mathbf{Q}_\ell) \rightarrow H_{\text{Zar}}^p(Y_s, \mathbf{Q}_\ell)$ is an isomorphism. The lemma follows. \square

Lemma 4.7. *Suppose that one of following conditions holds:*

- (1) *K is a local non-Archimedean field;*
- (2) *there is a finite extension K' of K such that $X \otimes_K K'$ is the generic fiber of a proper strictly semistable scheme over K'° .*

Then the map κ_X^p is injective for all $p \geq 0$.

Proof. Case (1) follows from [Ber00, Corollary 1.2]. Case (2) follows from Lemma 4.6, and the fact that the map $H^p(X^{\text{an}}, \mathbf{Q}_\ell) \rightarrow H^p((X \otimes_K K')^{\text{an}}, \mathbf{Q}_\ell)$ is injective. \square

5. MONODROMY MAPS FOR TROPICAL DOLBEAULT COHOMOLOGY

In this section, we introduce the conjecture on the isomorphism of tropical Dolbeault cohomology groups under monodromy maps. Then we prove our main results.

Let K be a non-Archimedean field. Let X be a separated scheme of finite type over K .

Definition 5.1 (Tropical Dolbeault cohomology). We define the *tropical Dolbeault cohomology* of X to be

$$H_{\text{trop}}^{p,q}(X) := H^{p,q}(X^{\text{an}}),$$

and the corresponding *tropical Hodge number* of X to be

$$h_{\text{trop}}^{p,q}(X) = \dim_{\mathbf{R}} H_{\text{trop}}^{p,q}(X).$$

We have the monodromy map

$$N_X = N_{X^{\text{an}}} : H_{\text{trop}}^{p,q}(X) \rightarrow H_{\text{trop}}^{p-1,q+1}(X)$$

for $p \geq 1$.

Conjecture 5.2. *Suppose that K is an algebraically closed non-Archimedean field such that \widetilde{K} is algebraic over a finite field. Let X be a proper smooth scheme over K . Then for $p \geq q \geq 0$, the (iterated) monodromy map*

$$N_X^{p-q} : H_{\text{trop}}^{p,q}(X) \rightarrow H_{\text{trop}}^{q,p}(X)$$

is an isomorphism.

Remark 5.3. Conjecture 5.2 does not hold for arbitrary algebraically closed non-Archimedean fields. In fact, let X_0 be the generic fiber of the scheme \mathfrak{X} in [BGS95, (6.1)], which is a geometrically connected projective smooth curve over $K_0 = \mathbf{C}((t))$. Take $K = \widehat{K_0^a}$ and $X = X_0 \otimes_{K_0} K$. Then one can show that $h_{\text{trop}}^{1,0}(X) = 0$ but $h_{\text{trop}}^{0,1}(X) = 2$. This also implies that the canonical pairing $H_{\text{trop}}^{1,0}(X) \times H_{\text{trop}}^{0,1}(X) \rightarrow H_{\text{trop}}^{1,1}(X) \xrightarrow{\text{Tr}} \mathbf{R}$ (where Tr is the integration map) is not a perfect pairing; in other words, the Poincaré duality fails.

Definition 5.4. Let X be a proper smooth scheme over a discrete non-Archimedean field K . A *strict semistable alteration* of X is a proper strictly semistable scheme \mathcal{X} over $K^\circ_{\mathcal{X}}$ for some finite extension $K_{\mathcal{X}}/K$ contained in K^a together with a proper dominant generically finite morphism $\phi: \mathcal{X}_\eta \rightarrow X$ over K , such that every irreducible component of $Y^{(r)}$ for $r \geq 0$ is geometrically irreducible.

Theorem 5.5. Let X be a proper smooth scheme over a discrete non-Archimedean field K and $p \geq 0$ an integer. Suppose that κ_X^p (4.4) is injective.

- (1) If $\log |K^\times| \subset \mathbf{Q}$, then the monodromy map $N_X^p: H_{\text{trop}}^{p,0}(X) \rightarrow H_{\text{trop}}^{0,p}(X)$ is rational, that is, it sends $H^0(X^{\text{an}}, \mathcal{I}_{X^{\text{an}}}^p)$ into $H^p(X^{\text{an}}, \mathbf{Q})$.
- (2) Suppose that for every strict semistable alteration \mathcal{X} of X (Definition 5.4), the map

$$N_{\mathcal{X}}^p: E_2^{-p,2p}(p) \rightarrow E_2^{p,0}$$

is injective. Then $N_X^p: H_{\text{trop}}^{p,0}(X) \rightarrow H_{\text{trop}}^{0,p}(X)$ is injective.

We need some preparation before the proof. The case $p = 0$ is trivial. So we assume $p \geq 1$. The following notation will be used later.

Notation 5.6. Let $h: T' \rightarrow T$ be a homomorphism of K -analytic torus. Then we denote by $h^b: T'_{\text{trop}} \rightarrow T_{\text{trop}}$ the induced linear map under tropicalization.

Let \mathcal{X} be a proper strictly semistable scheme over $K^\circ_{\mathcal{X}}$. We have the reduction map

$$\pi_{\mathcal{X}}: \mathcal{X}_\eta^{\text{an}} \rightarrow Y.$$

Let ω be an element of $H^0(X^{\text{an}}, \mathcal{I}_{X^{\text{an}}}^p)$. We say that a strict semistable alteration \mathcal{X} of X (with a morphism $\phi: \mathcal{X}_\eta \rightarrow X$) *presents* ω if for every irreducible component Y^i of Y , there exist

- an open subset U of $\mathcal{X}_\eta^{\text{an}}$ containing $\pi_{\mathcal{X}}^{-1}Y^i$,
- $c_l \in \mathbf{Q}$ for $1 \leq l \leq M$ with some integer $M = M_U \geq 1$,
- $f_{lk} \in \mathcal{O}_{\mathcal{X}_\eta^{\text{an}}}^*(U)$ for $1 \leq l \leq M$ and $1 \leq k \leq p$ satisfying $|f_{lk}| = 1$ on $\pi_{\mathcal{X}}^{-1}(Y^i \setminus Y^{[1]})$,

such that $\phi^{\text{an}*}\omega|_U = \tau \left(\sum_{l=1}^M c_l \{f_{l1}, \dots, f_{lp}\} \right)$. We call such data $(U, \{c_l\}, \{f_{lk}\})$ a *presentation* of ω on Y^i .

Now let Z be an irreducible component of Y^I for some $I = \{i_0 < \dots < i_p\}$ and $p \geq 1$. Choose a presentation $(U, \{c_l\}, \{f_{lk}\})$ of ω on Y^i . For $j = 1, \dots, p$, let $a_{lkj} \in \mathbf{Z}$ be the order of zero (or the negative order of pole) of f_{lk} along the connected component of $Y^{i_0} \cap Y^{i_j}$ containing Z .

Lemma 5.7. Let notation be as above. The rational number $\sum_{l=1}^M c_l \det(a_{lkj})_{k,j=1}^p$ depends only on ω and Z .

Proof. Let η_Z be the generic point of Z . Then $I(\eta_Z) = \{i_0 < \dots < i_p\}$ (Notation 4.2). We choose a semistable chart $(\mathcal{V}, \{g_0, \dots, g_p\})$ of η_Z (Construction 4.3). Put $V := \mathcal{V} \cap Y^{i_0}$, and replace U by $U \cap \mathcal{V}_\eta^{\text{an}}$ which is an open neighborhood of $\pi_{\mathcal{X}}^{-1}V$. We regard g_1, \dots, g_p as elements in $\mathcal{O}_{\mathcal{X}_\eta^{\text{an}}}^*(U)$. Then for $1 \leq j \leq p$, $|g_j| = 1$ on $\pi_{\mathcal{X}}^{-1}(V \setminus Y^{[1]})$ and the divisor associated to the reduction $\widetilde{g_j}$ on V is $V \cap Y^{i_j}$.

Note that the divisor associated to $\widetilde{f_{lk}}|_V$ is supported on $V \cap (\bigcup_{j=1}^p Y^{i_j})$. Put

$$f'_{lk} = f_{lk} \cdot \prod_{j=1}^p g_j^{-a_{lkj}}.$$

Then the reduction $\widetilde{f'_{lk}}$ is invertible on V . In other words, $|f'_{lk}| = 1$ on $\pi_{\mathcal{X}}^{-1}V$. Thus, $\tau(f'_{lk}) = 0$ on $\pi_{\mathcal{X}}^{-1}V$. An elementary calculation shows that

$$(5.1) \quad \phi^{\text{an}*}\omega|_{\pi_{\mathcal{X}}^{-1}V} = \tau\left(\sum_{l=1}^M c_l \{f_{l1}, \dots, f_{lp}\}\right) = \left(\sum_{l=1}^M c_l \det(a_{lkj})_{k,j=1}^p\right) \tau(\{g_1, \dots, g_p\}).$$

It is clear that $\tau(\{g_1, \dots, g_p\})$ is nonzero as V contains η_Z . The lemma then follows. \square

We denote the rational number in the above lemma by $\text{ord}_{\omega}(Z)$. The assignment $Z \mapsto \text{ord}_{\omega}(Z)$ gives rise to an element $\text{ord}_{\omega} \in H_{\text{ét}}^0(Y_s^{(p)}, \mathbf{Q}_{\ell})$. Denote by $H^0(X^{\text{an}}, \mathcal{T}_{X^{\text{an}}}^p)_{\mathcal{X}}$ the subset of $H^0(X^{\text{an}}, \mathcal{T}_{X^{\text{an}}}^p)$ consisting of ω which \mathcal{X} presents. Then it is easy to see that $H^0(X^{\text{an}}, \mathcal{T}_{X^{\text{an}}}^p)_{\mathcal{X}}$ is a \mathbf{Q} -subspace.

Lemma 5.8. *Let $\text{ord}: H^0(X^{\text{an}}, \mathcal{T}_{X^{\text{an}}}^p)_{\mathcal{X}} \rightarrow H_{\text{ét}}^0(Y_s^{(p)}, \mathbf{Q}_{\ell})$ be the map sending ω to ord_{ω} .*

- (1) *The map ord is injective.*
- (2) *The image of ord is contained in the subspace $E_2^{-p,2p}(p) \subset H_{\text{ét}}^0(Y_s^{(p)}, \mathbf{Q}_{\ell})$ (4.3).*

Proof. For (1), take an element $\omega \in H^0(X^{\text{an}}, \mathcal{T}_{X^{\text{an}}}^p)_{\mathcal{X}}$. Let $x \in \mathcal{X}_{\eta}^{\text{an}}$ be a point, and put $y = \pi_{\mathcal{X}}(x) \in Y$. Suppose that $I(y) = \{i_0 < \dots < i_r\}$ (Notation 4.2) for some $r \geq 0$, and choose a semistable chart $(\mathcal{V}, \{g_0, \dots, g_r\})$ of y (Construction 4.3). Put $V := \mathcal{V} \cap Y^{i_0}$.

Choose a presentation $(U, \{c_l\}, \{f_{lk}\})$ of ω on Y^{i_0} . We replace U by $U \cap \mathcal{V}_{\eta}^{\text{an}}$, and view g_1, \dots, g_r as in $\mathcal{O}_{\mathcal{X}_{\eta}^{\text{an}}}^*(U)$. Then there exist unique integers a_{lkj} such that if we put $f'_{lk} = \prod_{j=1}^r g_j^{a_{lkj}}$, then $|f'_{lk}| = |f_{lk}|$ on $\pi_{\mathcal{X}}^{-1}V$. Let f (resp. f') be the moment map $\pi_{\mathcal{X}}^{-1}V \rightarrow (\mathbf{G}_m^{\text{an}})^{Mp}$ induced by $\{f_{lk}\}$ (resp. $\{f'_{lk}\}$). Then $f_{\text{trop}} = f'_{\text{trop}}$. In particular,

$$\phi^{\text{an}*}\omega|_{\pi_{\mathcal{X}}^{-1}V} = \tau\left(\sum_{l=1}^M c_l \{f'_{l1}, \dots, f'_{lp}\}\right).$$

Let $g: \pi_{\mathcal{X}}^{-1}V \rightarrow (\mathbf{G}_m^{\text{an}})^r$ be the moment map induced by $\{g_1, \dots, g_r\}$. Then there exists a unique homomorphism $h: (\mathbf{G}_m^{\text{an}})^r \rightarrow (\mathbf{G}_m^{\text{an}})^{Mp}$ determined by integers $\{a_{lkj}\}$ such that $f_{\text{trop}} = (h \circ g)_{\text{trop}}$. Now if $\text{ord}_{\omega}(Z) = 0$ for every irreducible component Z of $Y^{[p]}$, then the pullback of $\tau\left(\sum_{l=1}^M c_l \{f'_{l1}, \dots, f'_{lp}\}\right)$ under h^b (Notation 5.6) is zero. In other words, we have $\phi^{\text{an}*}\omega|_{\pi_{\mathcal{X}}^{-1}V} = 0$. Thus the map $\text{ord}: H^0(X^{\text{an}}, \mathcal{T}_{X^{\text{an}}}^p)_{\mathcal{X}} \rightarrow H_{\text{ét}}^0(Y_s^{(p)}, \mathbf{Q}_{\ell})$ is injective since $\pi_{\mathcal{X}}^{-1}V$ is a neighborhood of x and x is arbitrary; (1) follows.

For (2), we need to show that $\delta_{p*}\text{ord}_{\omega} = \delta_p^*\text{ord}_{\omega} = 0$. Suppose that Z is an irreducible component of Y^I for some $I = \{i_0 < \dots < i_p\}$. For every permutation σ of the set $\{0, \dots, p\}$, we may define in the same way a rational number $\text{ord}_{\omega}^{\sigma}(Z)$ by replacing i_j by $i_{\sigma(j)}$. If $\sigma(0) = 0$, then we let $\epsilon(\sigma) \in \{\pm 1\}$ be the signature of the permutation $\sigma|_{\{1, \dots, p\}}$. If $\sigma(0) \neq 0$, then we let $\epsilon(\sigma) \in \{\pm 1\}$ be the negative of the signature of the permutation from $\{1, \dots, 0, \dots, p\}$ ($\sigma(0)$ is replaced by 0) to $\{\sigma(1), \dots, \sigma(p)\}$. Then the proof of Lemma 5.7 implies that

$$(5.2) \quad \text{ord}_{\omega}^{\sigma}(Z) = \epsilon(\sigma) \cdot \text{ord}_{\omega}(Z).$$

We start from $\delta_{p*}\text{ord}_{\omega}$. Fix an irreducible component W of Y^J for some $J = \{i_0 < \dots < i_{p-1}\}$. For $1 \leq j \leq p-1$, let W_j be the unique irreducible component of $Y^{i_0} \cap Y^{i_j}$ that contains W . Choose a presentation $(U, \{c_l\}, \{f_{lk}\})$ of ω on Y^{i_0} . By linear algebra, it is easy to see that we may choose f_{lk} such that $\text{ord}_{W_j}(f_{lk}) = 0$ if $j < k$. In particular, the restriction of $\widetilde{f'_{lp}}$ on W is a nonzero rational function, which we denote by f_{lp}^W . Put $b_{l,j} = \text{ord}_{W_j}(\widetilde{f'_{lj}})$ for $1 \leq j \leq p-1$. Let Z be an irreducible component of $W \cap Y^{[p]}$. Then there is a unique

element $i_p \in \{1, \dots, m\} \setminus \{i_0, \dots, i_{p-1}\}$ such that $Z \subset W \cap Y^{i_p}$. Let $0 \leq \epsilon_Z \leq p$ be the integer such that there are exactly ϵ_Z elements in $\{i_0, \dots, i_{p-1}\}$ that are greater than i_p . Then by (5.2), we have

$$\text{ord}_\omega(Z) = (-1)^{\epsilon_Z} \sum_{l=1}^M c_l b_{l,1} \cdots b_{l,p-1} \text{ord}_Z(f_{l_p}^W).$$

Therefore, we have the equality

$$(-1)^{p-\epsilon_Z} \sum_{Z \subset W} \text{ord}_\omega(Z) Z = (-1)^p \sum_{l=1}^M c_l b_{l,1} \cdots b_{l,p-1} \text{div}(f_{l_p}^W)$$

of divisors on W . Thus $\delta_{p*} \text{ord}_\omega = 0$ by (4.2).

Now we consider $\delta_p^* \text{ord}_\omega$. Let W be an irreducible component of Y^J for some $J = \{i_0 < \dots < i_{p+1}\}$. For $0 \leq j \leq p+1$, let Z_j be the unique irreducible component of $Y^{J \setminus \{i_j\}}$ containing W . Let $(\mathcal{V}, \{g_0, \dots, g_{p+1}\})$ be a semistable chart of the generic point of W (Construction 4.3). Put $V = \mathcal{V} \cap Y$ and $V_j = V \setminus Y^{i_j}$. From the proof for (1), we know that

$$\phi^{\text{an}*} \omega|_{\pi_X^{-1}V} = \sum_{\alpha} c_{\alpha} \tau(\{g_{\alpha(1)}, \dots, g_{\alpha(p)}\})$$

for some $c_{\alpha} \in \mathbf{Q}$, where the sum is taken over all strictly increasing maps $\alpha: \{1, \dots, p\} \rightarrow \{0, \dots, p+1\}$. We take such a map α . Now let $\alpha_0 < \alpha_1$ be the two integers in $\{0, \dots, p+1\}$ not in the image of α . We have three cases:

- If $j \in \text{im}(\alpha)$, then $\tau(\{g_{\alpha(1)}, \dots, g_{\alpha(p)}\})|_{\pi_X^{-1}V_j} = 0$.
- If $j = \alpha_0$, then $\tau(\{g_{\alpha(1)}, \dots, g_{\alpha(p)}\})|_{\pi_X^{-1}V_j} = (-1)^{\alpha_1-1} \tau(\{g_{\alpha'(1)}, \dots, g_{\alpha'(p)}\})$, where α' is the unique map whose image does not include α_0 and $\min(\{0, \dots, p+1\} \setminus \{\alpha_0\})$.
- If $j = \alpha_1$, then $\tau(\{g_{\alpha(1)}, \dots, g_{\alpha(p)}\})|_{\pi_X^{-1}V_j} = (-1)^{\alpha_0} \tau(\{g_{\alpha'(1)}, \dots, g_{\alpha'(p)}\})$, where α' is the unique map whose image does not include $\{0, \alpha_1\}$.

By (5.1), we have

$$\text{ord}_\omega(Z_j) = \sum_{\alpha, \alpha_0=j} c_{\alpha} (-1)^{\alpha_1-1} + \sum_{\alpha, \alpha_1=j} c_{\alpha} (-1)^{\alpha_0}.$$

However, by (4.1), we have

$$\delta_p^* \omega|_W = \sum_{j=0}^{p+1} (-1)^j \text{ord}_\omega(Z_j) = \sum_{\alpha} c_{\alpha} (-1)^{\alpha_0+\alpha_1-1} + c_{\alpha} (-1)^{\alpha_0+\alpha_1} = 0.$$

Thus (2) follows. \square

The map ord in Lemma 5.8 is inspired and closely related to the construction in [Liu17, §4]. The following lemma is the key step to the proof of Theorem 5.5. It also justify the terminology *monodromy map* for N_X in Definition 3.1 and Definition 5.1.

Lemma 5.9. *Let \mathcal{X} be a strict semistable alteration of X , with $\phi: \mathcal{X}_\eta \rightarrow X$. Suppose $\log |\varpi| = -1$ for a uniformizer ϖ of K . Then the image of the restriction of $N_X^p: H_{\text{trop}}^{p,0}(X) \rightarrow H_{\text{trop}}^{0,p}(X)$ to $H^0(X^{\text{an}}, \mathcal{T}_{X^{\text{an}}}^p)_{\mathcal{X}}$ is contained in $H^p(X^{\text{an}}, \mathbf{Q})$. Moreover, the following diagram*

$$(5.3) \quad \begin{array}{ccc} H^0(X^{\text{an}}, \mathcal{T}_{X^{\text{an}}}^p)_{\mathcal{X}} & \xrightarrow{\text{ord}} & E_2^{-p,2p}(p) \\ N_X^p \downarrow & & \downarrow N_{\mathcal{X}}^p \\ H^p(X^{\text{an}}, \mathbf{Q}) & \longrightarrow & E_2^{p,0} \end{array}$$

commutes. Here, the bottom map is the composition of

- the multiplication map $(-1)^{p(p+1)/2}: H^p(X^{\text{an}}, \mathbf{Q}) \rightarrow H^p(X^{\text{an}}, \mathbf{Q}_\ell)$,
- the map $\kappa_X^p: H^p(X^{\text{an}}, \mathbf{Q}_\ell) \rightarrow H_{\text{ét}}^p(X_a, \mathbf{Q}_\ell)$ (4.4) (which is assumed to be injective), and
- the pullback map $\phi^*: H_{\text{ét}}^p(X_a, \mathbf{Q}_\ell) \rightarrow H_{\text{ét}}^p(\mathcal{X}_{\eta,a}, \mathbf{Q}_\ell)$.

The image of the bottom map is contained in $E_2^{p,0}$ by Lemma 4.6.

Proof. To simplify notation, put $\mathcal{A}_{\bullet}^{p,q,\text{cl}} := \ker[d'': \mathcal{A}_{\bullet}^{p,q} \rightarrow \mathcal{A}_{\bullet}^{p,q+1}]$. Take an element $\omega \in H^0(X^{\text{an}}, \mathcal{T}_{X^{\text{an}}}^p)_{\mathcal{X}}$. By Lemma 3.2 (1), $N_X^p(\omega)$ is represented by the Dolbeault representative $p!J\omega \in H^0(X^{\text{an}}, \mathcal{A}_{X^{\text{an}}}^{0,p,\text{cl}})$.

Suppose that (5.3) commutes. By the projection formula, we have $\phi_* \circ \phi^* = [K_{\mathcal{X}}(\mathcal{X}_\eta) : K(X)] \neq 0$. Thus ϕ^* is injective. Choose an embedding $\mathbf{R} \hookrightarrow \mathbf{Q}_\ell^a$. Then the map $H^p(X^{\text{an}}, \mathbf{R}) \rightarrow E_2^{p,0} \otimes_{\mathbf{Q}_\ell} \mathbf{Q}_\ell^a$ is injective. Thus $N_X^p(\omega) \in H^p(X^{\text{an}}, \mathbf{Q})$.

Now we focus on the commutativity. For $1 \leq i \leq m$, put $U^i = \pi_{\mathcal{X}}^{-1}Y^i$, and choose a presentation $(U^i, \{c_l^i\}, \{f_{lk}^i\} \mid 1 \leq l \leq M_i, 1 \leq k \leq p)$ of ω on Y^i . For every subset $I \subset \{1, \dots, m\}$, put $U^I = \bigcap_{i \in I} U^i = \pi_{\mathcal{X}}^{-1}Y^I$. Since $\pi_0(U^I) = \pi_0(Y^I)$, the assignment $Z \mapsto \text{ord}_\omega(Z)$ gives rise to a Čech p -cocycle θ_ω for the sheaf \mathbf{Q} with respect to the ordered open covering $\underline{U} = \{U^1, \dots, U^m\}$ of \mathcal{X}^{an} . Since $\delta_p^* \text{ord}_\omega = 0$ by Lemma 5.8 (2), θ_ω is closed hence gives rise to a class in $H^p(\underline{U}, \mathbf{Q})$, whose image $[\theta_\omega] \in H^p(\mathcal{X}_\eta^{\text{an}}, \mathbf{Q})$ coincides with $(\kappa_{\mathcal{X}_\eta}^p)^{-1}(N_{\mathcal{X}}^p(\text{ord}_\omega))$ where $\kappa_{\mathcal{X}_\eta}^p: H^p(\mathcal{X}_\eta^{\text{an}}, \mathbf{Q}_\ell) \xrightarrow{\sim} E_2^{p,0}$ is the isomorphism in Lemma 4.6. Therefore by Lemma 3.2 (1), the commutativity of (5.3) is equivalent to that $(-1)^{p(p+1)/2}p!J\omega$ is a Dolbeault representative of $[\theta_\omega]$.

Let us recall the construction of Dolbeault representatives. For an abelian sheaf \mathcal{F} on $\mathcal{X}_\eta^{\text{an}}$ and $r \geq 0$, denote by

$$C^r(\underline{U}, \mathcal{F}) = \bigoplus_{|I|=r+1} \Gamma(U^I, \mathcal{F})$$

the abelian group of Čech r -cocycles for \mathcal{F} with respect to \underline{U} . We have a coboundary map $\delta: C^r(\underline{U}, \mathcal{F}) \rightarrow C^{r+1}(\underline{U}, \mathcal{F})$. By the Poincaré lemma ([Jel16a, Corollary 4.6]), we have a short exact sequence

$$0 \rightarrow \mathcal{A}_{\mathcal{X}_\eta^{\text{an}}}^{0,r,\text{cl}} \rightarrow \mathcal{A}_{\mathcal{X}_\eta^{\text{an}}}^{0,r} \xrightarrow{d''} \mathcal{A}_{\mathcal{X}_\eta^{\text{an}}}^{0,r+1,\text{cl}} \rightarrow 0.$$

If we have elements $\theta_r \in C^r(\underline{U}, \mathcal{A}_{\mathcal{X}_\eta^{\text{an}}}^{0,p-r-1})$ for $0 \leq r \leq p-1$ satisfying

$$(5.4) \quad d''\theta_0 = J\omega, \quad d''\theta_1 = \delta\theta_0, \quad \dots \quad d''\theta_{p-1} = \delta\theta_{p-2}, \quad \frac{(-1)^{p(p+1)/2}}{p!}\theta_\omega = \delta\theta_{p-1},$$

then $(-1)^{p(p+1)/2}p!J\omega$ is a Dolbeault representative of $[\theta_\omega]$ and (5.3) commutes. Here, we regard $J\omega$ as an element in $C^0(\underline{U}, \mathcal{A}_{X^{\text{an}}}^{0,p})$.

The remaining proof will be dedicated to the construction of θ_r . For $r \geq 0$, denote by \mathcal{D}^r the set of irreducible components of $Y^{[r]}$.

Step 1. For $Z \in \mathcal{D}^r$, put $U_Z = \pi_{\mathcal{X}}^{-1}Z$, $M_Z = \sum_{i \in I(Z)} M_i$ and let

$$f_Z: U_Z \rightarrow (\mathbf{G}_m^{\text{an}})^{M_Z p}$$

be the moment map given by invertible functions $\{f_{lk}^i \mid i \in I(Z), 1 \leq l \leq M_i, 1 \leq k \leq p\}$ on U . Put $U_Z^\circ = \pi_{\mathcal{X}}^{-1}(Z \setminus Y^{[r+1]})$. The image of the tropicalization map $f_{Z,\text{trop}}: U_Z \rightarrow \mathbf{R}^{M_Z p}$, denoted by C_Z , is canonically a simplicial complex, induced by the reduced normal crossing

divisor $Z \cup Y^{[r+1]}$ of Z , with the unique minimal simplex $\Delta_Z := f_{Z,\text{trop}}(U_Z^\circ)$. We now define the *barycenter* P_Z of Δ_Z .

Let η_Z be the generic point of Z . Let $(\mathcal{V}, \{g_0, \dots, g_r\})$ be a semistable chart at η_Z . We let $g_Z: U_Z \cap \mathcal{V}_\eta^{\text{an}} \rightarrow (\mathbf{G}_m^{\text{an}})^{r+1}$ be the moment map induced by $\{g_0, \dots, g_r\}$. Then the image of $g_{Z,\text{trop}}$ is the standard (open) simplex Δ° in \mathbf{R}^{r+1} , which is $\{(x_0, \dots, x_r) \in \mathbf{R}^{r+1} | x_0 + \dots + x_r = 1, x_i > 0\}$. Similar to the proof of Lemma 5.8, we have unique integers a_{lkj}^i such that

$$f_{lk}^i \cdot \prod_{j=0, j \neq i}^r g_j^{-a_{lkj}^i}$$

has normal 1 on $\pi_{\mathcal{X}}^{-1}(Z \cap \mathcal{V}) \subset U_Z^\circ$. Then there exists a unique homomorphism

$$(5.5) \quad h_Z: (\mathbf{G}_m^{\text{an}})^{r+1} \rightarrow (\mathbf{G}_m^{\text{an}})^{M_Z p}$$

determined by integers $\{a_{lkj}^i\}$ such that $f_{Z,\text{trop}}|_{\pi_{\mathcal{X}}^{-1}(Z \cap \mathcal{V})} = (h_Z \circ g_Z)_{\text{trop}}$. Then we have

$$h_Z^b(\Delta^\circ) = f_{Z,\text{trop}}(\pi_{\mathcal{X}}^{-1}(Z \cap \mathcal{V})) \subset \Delta_Z.$$

We define P_Z to be the image under h_Z^b (Notation 5.6) of the barycenter of Δ° , which is $\{(\frac{1}{r+1}, \dots, \frac{1}{r+1})\}$. It is easy to see that the point P_Z does not depend on the choice of the semistable chart of η_Z (and in fact, $h_Z^b(\Delta^\circ) = \Delta_Z$). To summarize, we have $P_Z \in \Delta_Z \subset C_Z \subset \mathbf{R}^{M_Z p}$.

Step 2. Take another irreducible component $Z' \in \mathcal{D}^{r'}$ such that $Z \subset Z'$. In particular, we have $r' \leq r$ and $I(Z') \subset I(Z)$. Let

$$(5.6) \quad h_{Z,Z'}: (\mathbf{G}_m^{\text{an}})^{M_Z p} \rightarrow (\mathbf{G}_m^{\text{an}})^{M_{Z'} p}$$

be the canonical projection by forgetting components with $i \in I(Z) \setminus I(Z')$. Then the following diagram

$$\begin{array}{ccc} U_Z & \xrightarrow{f_Z} & (\mathbf{G}_m^{\text{an}})^{M_Z p} \\ i_{Z,Z'} \downarrow & & \downarrow h_{Z,Z'} \\ U_{Z'} & \xrightarrow{f_{Z'}} & (\mathbf{G}_m^{\text{an}})^{M_{Z'} p} \end{array}$$

commutes, where $i_{Z,Z'}$ is the natural inclusion map. It induces, for every $q \geq 0$, the following commutative diagram

$$\begin{array}{ccc} \mathcal{A}_{C_{Z'}}^{0,q}(C_{Z'}) & \xrightarrow{\iota_{Z'}} & \mathcal{A}_{\mathcal{X}_\eta^{\text{an}}}^{0,q}(U_{Z'}) \\ (h_{Z,Z'}^b)^* \downarrow & & \downarrow i_{Z,Z'}^* \\ \mathcal{A}_{C_Z}^{0,q}(C_Z) & \xrightarrow{\iota_Z} & \mathcal{A}_{\mathcal{X}_\eta^{\text{an}}}^{0,q}(U_Z) \end{array}$$

where ι_Z and $\iota_{Z'}$ are natural maps from the definition of $\mathcal{A}_{\mathcal{X}_\eta^{\text{an}}}^{0,q}$.

On the other hand, we have a canonical isomorphism

$$C^r(\underline{U}, \mathcal{A}_{\mathcal{X}_\eta^{\text{an}}}^{0,q}) \simeq \bigoplus_{Z \in \mathcal{D}^r} \mathcal{A}_{\mathcal{X}_\eta^{\text{an}}}^{0,q}(U_Z),$$

which induces a canonical map

$$\iota^r: \bigoplus_{Z \in \mathcal{D}^r} \mathcal{A}_{C_Z}^{0,q}(C_Z) \rightarrow C^r(\underline{U}, \mathcal{A}_{\mathcal{X}_\eta^{\text{an}}}^{0,q})$$

for $r \geq 0$, and there is a similarly defined coboundary map

$$(5.7) \quad \delta: \bigoplus_{Z \in \mathcal{D}^r} \mathcal{A}_{C_Z}^{0,q}(C_Z) \rightarrow \bigoplus_{Z' \in \mathcal{D}^{r+1}} \mathcal{A}_{C_{Z'}}^{0,q}(C_{Z'})$$

using $(h_{Z,Z'}^b)^*$ as restriction maps, such that it is compatible with the coboundary map for $C^\bullet(\underline{U}, \mathcal{A}_{\mathcal{X}_\eta^{\text{an}}}^{0,q})$ under ι^\bullet .

Moreover, since C_Z is star-shaped with respect to P_Z , we have the star-shape integration map

$$\mathcal{I}_{P_Z}'': \mathcal{A}_{C_Z}^{0,q}(C_Z) \rightarrow \mathcal{A}_{C_Z}^{0,q-1}(C_Z)$$

for $q \geq 1$ (see §6 for a formula on the standard simplex). We put

$$\mathcal{I}'' := \bigoplus \mathcal{I}_{P_Z}'': \bigoplus_{Z \in \mathcal{D}^r} \mathcal{A}_{C_Z}^{0,q}(C_Z) \rightarrow \bigoplus_{Z \in \mathcal{D}^r} \mathcal{A}_{C_Z}^{0,q-1}(C_Z).$$

Step 3. Note that $\mathcal{D}^0 = \{Y_i \mid 1 \leq i \leq m\}$. Define $\vartheta_0 \in \bigoplus_{Z \in \mathcal{D}^0} \mathcal{A}_{C_Z}^{0,p-1}(C_Z)$ by the formula

$$\vartheta_0(Y^i) = \mathcal{I}_{P^{Y^i}}'' \left(\sum_{l=1}^{M_i} c_l^i d'' x_{l1}^i \wedge \cdots \wedge d'' x_{lp}^i \right),$$

where x_{lk}^i is the standard coordinates on $\mathbf{R}^{M_i p}$. For $1 \leq r \leq p-1$, we define

$$\vartheta_r = \mathcal{I}''(\delta \vartheta_{r-1}) \in \bigoplus_{Z \in \mathcal{D}^r} \mathcal{A}_{C_Z}^{0,p-r-1}(C_Z).$$

We claim that

$$(5.8) \quad d'(\delta \vartheta_{r-1}) = d''(\delta \vartheta_{r-1}) = 0$$

for $1 \leq r \leq p-1$, and

$$(5.9) \quad \delta \vartheta_{p-1} = \frac{(-1)^{p(p+1)/2}}{p!} \theta_w.$$

We leave the proof to the next step. Assuming these, then $d'' \vartheta_r = \delta \vartheta_{r-1}$ as $d''(\delta \vartheta_{r-1}) = 0$. Finally, we put $\theta_r = \iota^r \vartheta_r \in C^r(\underline{U}, \mathcal{A}_{\mathcal{X}_\eta^{\text{an}}}^{0,p-r-1})$ for $0 \leq r \leq p-1$. Then they satisfies (5.4).

Step 4. We have to verify (5.8) and (5.9) for $(\delta \vartheta_{r-1})(Z) \in \mathcal{A}_{C_Z}^{0,p-r}(C_Z)$ for every $Z \in \mathcal{D}^r$. Without lost of generality, we assume that X has dimension n . For each fixed $Z \in \mathcal{D}^r$, it suffices to consider the restriction of $(\delta \vartheta_{r-1})(Z)$ to maximal (open) cells of C_Z , which are all of the form $f_{Z,\text{trop}}(U_z)$ where $z \in Z$ is a closed point that belongs to \mathcal{D}^n .

The ideal is to reverse the consideration. We fix a closed point $z \in \mathcal{D}^n$ and consider the restriction of $(\delta \vartheta_{r-1})(Z)$ to $f_{Z,\text{trop}}(U_z)$ for all $1 \leq r \leq p$ and all $Z \in \mathcal{D}^r$ such that $z \in Z$. We fix a semistable chart $(\mathcal{V}, \{g_0, \dots, g_n\})$ at z . Then $\mathcal{V}_\eta^{\text{an}}$ contains U_z . We have the moment map $g: \mathcal{V}_\eta^{\text{an}} \rightarrow (\mathbf{G}_m^{\text{an}})^{n+1}$ and $g_{\text{trop}}: \mathcal{V}_\eta^{\text{an}} \rightarrow \mathbf{R}^{n+1}$ so that $g_{\text{trop}}(U_z)$ is the standard open simplex Δ° in \mathbf{R}^{n+1} . Note that for every subset $I \subset I(z)$ of cardinality $r+1$, there is a unique element $Z \in \mathcal{D}^r$ that contains z , which we denote by Z_I . For every I as above, put

$$h_I := h_{z,Z_I} \circ h_z: (\mathbf{G}_m^{\text{an}})^{n+1} \rightarrow (\mathbf{G}_m^{\text{an}})^{M_{Z_I} p}$$

where h_z and h_{z,Z_I} are defined in (5.5) and (5.6), respectively. Then $f_{Z,\text{trop}}(U_z)$ is the image of Δ° under h_I^b . Define the composite map

$$h_I^\dagger: \mathcal{A}_{C_{Z_I}}^{0,q}(C_{Z_I}) \xrightarrow{(h_I^b)^*} \mathcal{A}_{\Delta^\circ}^{0,q}(\Delta^\circ) \rightarrow \Omega^q(\Delta)$$

where the second map is simply regarding a $(0, q)$ -superform as an ordinary q -form (see §6 for notation concerning Δ). Since h_I^\dagger is affine and $h_I(P^I) = P_{Z^I}$, the following diagram

$$\begin{array}{ccc} \mathcal{A}_{C_{Z^I}}^{0,q}(C_{Z^I}) & \xrightarrow{h_I^\dagger} & \Omega^q(\Delta) \\ \mathcal{I}_{P_{Z^I}}'' \downarrow & & \downarrow \mathcal{I}_{P^I} \\ \mathcal{A}_{C_{Z^I}}^{0,q-1}(C_{Z^I}) & \xrightarrow{h_I^\dagger} & \Omega^{q-1}(\Delta) \end{array}$$

commutes. Moreover, we have the following commutative diagram

$$\begin{array}{ccc} \bigoplus_{Z \in \mathcal{D}^r} \mathcal{A}_{C_Z}^{0,q}(C_Z) & \xrightarrow{h_r} & C^r(\Omega^q(\Delta)) \\ \delta \downarrow & & \downarrow \delta \\ \bigoplus_{Z' \in \mathcal{D}^{r+1}} \mathcal{A}_{C_{Z'}}^{0,q}(C_{Z'}) & \xrightarrow{h_{r+1}} & C^{r+1}(\Omega^q(\Delta)) \end{array}$$

concerning coboundary maps. Here, δ on the left-hand side is (5.7); δ on the right-hand side is defined in §6; and h_r is the composition of the projection

$$\bigoplus_{Z \in \mathcal{D}^r} \mathcal{A}_{C_Z}^{0,q}(C_Z) \rightarrow \bigoplus_{I \subset I(z), |I|=r+1} \mathcal{A}_{C_{Z^I}}^{0,q}(C_{Z^I})$$

and $\bigoplus_{I \subset I(z), |I|=r+1} h_I^\dagger$.

Take $\beta_0 = h_0(d''\vartheta_0)$. Then β_0 comes from a unique element $\beta \in \Omega^p(\Delta)$ as in Proposition 6.1. By the above discussion, we have

$$\beta_r = h_r(\delta\vartheta_{r-1})$$

for $1 \leq r \leq p$, where β_r is defined as in Proposition 6.1. In particular, (5.8) (resp. (5.9)) holds when restricted to $f_{Z, \text{trop}}(U_z)$ by Proposition 6.1 (1) (resp. (2)). Since z is arbitrary, (5.8) and (5.9) hold, and the lemma follows. \square

Lemma 5.10. *We have $H^0(X^{\text{an}}, \mathcal{T}_{X^{\text{an}}}^p) = \bigcup_{\mathcal{X}} H^0(X^{\text{an}}, \mathcal{T}_{X^{\text{an}}}^p)_{\mathcal{X}}$, where the union is taken over all strict semistable alterations \mathcal{X} of X .*

Proof. Let ω be an element in $H^0(X^{\text{an}}, \mathcal{T}_{X^{\text{an}}}^p)$. Since X is proper, we may choose a finite open covering \underline{U} of X^{an} such that for every $U \in \underline{U}$, there are $c_l \in \mathbf{Q}$ for $1 \leq l \leq M$ with some integer $M \geq 1$, and $f_{lk} \in \mathcal{O}_{X^{\text{an}}}^*(U)$ for $1 \leq l \leq M$ and $1 \leq k \leq p$, such that $\omega|_U = \tau \left(\sum_{l=1}^M c_l \{f_{l1}, \dots, f_{lp}\} \right)$.

By [Pay09, Theorem 4.2], taking blow-ups, and possibly taking a finite extension K'/K inside K^a , we have a (proper flat) integral model \mathcal{X}_0 of $X \otimes_K K'$ such that if $Y_0^1, \dots, Y_0^{m_0}$ are all reduced irreducible components of $\mathcal{X}_0 \otimes_{K'} \widetilde{K'}$, then the covering $\{\pi_{\mathcal{X}_0}^{-1} Y_0^i \mid i = 1, \dots, m_0\}$ refines \underline{U} . By [dJ96, Theorem 8.2], we have a proper strictly semistable scheme \mathcal{X} over $K_{\mathcal{X}}^\circ$, where $K_{\mathcal{X}}/K'$ is a finite extension inside K^a , with a proper dominant generically finite morphism $\mathcal{X} \rightarrow \mathcal{X}_0$. Replacing $K_{\mathcal{X}}$ by a finite unramified extension inside K^a , we may assume that every irreducible component of $Y^{(p)}$ for $p \geq 0$ is geometrically irreducible. Therefore, $(\mathcal{X}, \phi: \mathcal{X}_\eta \rightarrow X)$ is a strict semistable alteration of X such that $\{\pi_{\mathcal{X}}^{-1} Y^i \mid i = 1, \dots, m\}$ refines $(\phi^{\text{an}})^{-1} \underline{U}$.

Now for an arbitrary irreducible component Y^i of Y , take an open subset $U \in \underline{U}$ such that $\pi_{\mathcal{X}}^{-1} Y^i \subset (\phi^{\text{an}})^{-1} U$. For every f_{lk} as above, we may choose an element $a_{lk} \in K^*$ such

that $|a_{lk}\phi^{\text{an}*}f_{lk}| = 1$ on $\pi_{\mathcal{X}}^{-1}(Y^i \setminus Y^{[1]})$. Thus $((\phi^{\text{an}})^{-1}U, \{c_l\}, \{a_{lk}\phi^{\text{an}*}f_{lk}\})$ is a presentation of ω on Y^i . Therefore, ω belongs to $H^0(X^{\text{an}}, \mathcal{T}_{X^{\text{an}}}^p)_{\mathcal{X}}$. \square

Proof of Theorem 5.5. Part (1) is a consequence of Lemmas 5.9 and 5.10. For (2), it is equivalent to check that the map $N_X^p: H^0(X^{\text{an}}, \mathcal{T}_{X^{\text{an}}}^p) \rightarrow H^p(X^{\text{an}}, \mathbf{Q})$ is injective. Let ω be an element of $H^0(X^{\text{an}}, \mathcal{T}_{X^{\text{an}}}^p)$. By Lemma 5.10, we may assume that $\omega \in H^0(X^{\text{an}}, \mathcal{T}_{X^{\text{an}}}^p)_{\mathcal{X}}$ for a strict semistable alterations \mathcal{X} of X . Then it follows from Lemmas 5.8 and 5.9 as we assume that $N_{\mathcal{X}}^p$ is injective. \square

Corollary 5.11. *Let X_0 be a proper smooth scheme over K_0 that is isomorphic to $k((t))$ for k either a finite field or a field of characteristic zero. Let K be a closed subfield of \widehat{K}_0^{a} containing K_0 . Then the monodromy map*

$$N_X^p: H_{\text{trop}}^{p,0}(X) \rightarrow H_{\text{trop}}^{0,p}(X)$$

is injective, where $X = X_0 \otimes_{K_0} K$. In particular, we have $h_{\text{trop}}^{p,0}(X) < \infty$ and $h_{\text{trop}}^{p-q,q}(X) \geq h_{\text{trop}}^{p,0}(X)$ for $0 \leq q \leq p$.

Proof. Suppose first K/K_0 is a finite extension. By Lemma 4.7 and resolution of singularity when $\text{char } k = 0$, the map κ_X^p is injective. Then the corollary follows from Theorem 5.5 and Proposition 4.5 (1).

In general, since X_0 is proper, we have that

$$H_{\text{trop}}^{p,0}(X) = \bigcup_{K_0 \subset K' \subset K} H_{\text{trop}}^{p,0}(X_0 \otimes_{K_0} K')$$

where the union is taken over all finite extensions K'/K_0 contained in K . By [Ber99, Theorem 10.1], we know that $H_{\text{trop}}^{p,0}(X_0 \otimes_{K_0} K') \simeq H^p(X_0^{\text{an}} \otimes_{K_0} K', \mathbf{R})$ stabilizes when K' increases. Therefore, the injectivity of N_X^p is reduced to the case where K/K_0 is finite. \square

In the case where $p = 1$, we have the following stronger result over certain non-Archimedean fields.

Theorem 5.12. *Let K_0 be a local non-Archimedean field, and X_0 a proper smooth scheme over K_0 that admits a proper strictly semistable model over K_0° . Then the monodromy map*

$$N_X: H_{\text{trop}}^{1,0}(X) \rightarrow H_{\text{trop}}^{0,1}(X)$$

is an isomorphism, where $X = X_0 \otimes_{K_0} \widehat{K}_0^{\text{a}}$.

Proof. Since X_0 admits a proper strictly semistable model over K_0° , it is well-known (see for example [Sai03, Lemma 1.11]) that $X_0 \otimes_{K_0} K$ admits a proper strictly semistable model over K° for any finite extension of K of K_0 . Replacing K by a finite unramified extension, we may assume that X_0 admits a strict semistable alteration \mathcal{X} over K° such that $\phi: \mathcal{X}_{\eta} \rightarrow X_0$ induces an isomorphism $\mathcal{X}_{\eta} \xrightarrow{\sim} X_0 \otimes_{K_0} K$. We then have an infinite sequence of successive finite field extensions $K_0 \subset K_1 \subset \dots$ contained in \widehat{K}_0^{a} such that for $i \geq 1$, X_0 admits a strict semistable alteration \mathcal{X}_i over K_i° with $\phi: \mathcal{X}_{i,\eta} \xrightarrow{\sim} X_i$ being an isomorphism where $X_i := X_0 \otimes_{K_0} K_i$.

By the similar argument in the proof of Corollary 5.11, it suffices to show that

$$(5.10) \quad N_{X_i}: H_{\text{trop}}^{1,0}(X_i) \rightarrow H_{\text{trop}}^{0,1}(X_i)$$

is an isomorphism for every $i \geq 1$. By Theorem 5.5 (2), Lemma 4.7 and Proposition 4.5 (3), we know that (5.10) is injective. Thus if $h_{\text{trop}}^{1,0}(X_i) \geq h_{\text{trop}}^{0,1}(X_i)$, then (5.10) is an isomorphism.

We fix such an index i and suppress from notation. In particular, now we have $X = X_0 \otimes_{K_0} K$ with K a finite extension of K_0 . Let \mathcal{X} be a strict semistable alteration of X with an isomorphism $\phi: \mathcal{X}_\eta \xrightarrow{\sim} X$. We identify X with \mathcal{X}_η . By Lemma 4.6 and Proposition 4.5 (3), the composite map

$$N_{\mathcal{X}}^{-1} \circ \kappa_X^1: H^1(X^{\text{an}}, \mathbf{Q}_\ell) \rightarrow E_2^{-1,2}(1)$$

is an isomorphism, under which the image of $H^1(X^{\text{an}}, \mathbf{Q})$ is contained in $E_2^{-1,2}(1) \cap H_{\text{ét}}^0(Y_s^{(1)}, \mathbf{Q})$, which we denote by $E_{\mathcal{X}}$. The theorem will follow if we can construct an injective (linear) map

$$T_{\mathcal{X}}: E_{\mathcal{X}} \rightarrow H^0(X^{\text{an}}, \mathcal{L}_{X^{\text{an}}}^1).$$

For every point $y \in Y$, we fix a semistable chart $(\mathcal{V}_y, \{g_0^y, \dots, g_{p(y)}^y\})$ at y (Construction 4.3). We can write an element in $E_{\mathcal{X}}$ in the form

$$D = \frac{1}{q} \sum_{Z \in \mathcal{D}^1} a(Z) Z$$

for some integers $a(Z)$, $q > 0$, and the sum is taken over \mathcal{D}^1 , the set of all irreducible components of $Y^{[1]}$. As $\delta_{1*} D = 0$ (4.3), the divisor $(qD) \cap Y^i$ on Y^i is cohomologically trivial for every irreducible component Y^i of Y ($1 \leq i \leq m$). Then there exists some integer $q_i > 0$ such that $(q_i q D) \cap Y^i$ is algebraically equivalent to zero; and in particular $\mathcal{O}_{Y^i}((q_i q D) \cap Y^i)$ is an element in $\text{Pic}_{Y^i/\widetilde{K}}^0(\widetilde{K})$. Since $\text{Pic}_{Y^i/\widetilde{K}}^0$ is a projective scheme over the *finite* field \widetilde{K} , one may replace q_i by some integer multiple such that $\mathcal{O}_{Y^i}((q_i q D) \cap Y^i)$ is a trivial line bundle. Replacing q by $q \prod_{i=1}^m q_i$, we may assume that for every i , there is a rational function $f_i \in \widetilde{K}(Y^i)$ such that its divisor is exactly $(qD) \cap Y^i$.

For every point y of Y , we are going to construct an open neighborhood $U_y \subset X^{\text{an}}$ of $\pi_{\mathcal{X}}^{-1} y$ and an element $\omega_y \in \mathcal{L}_{X^{\text{an}}}^1(U_y)$. For every nonempty subset I of $I(y)$ (Notation 4.2), let Z_y^I be the unique irreducible component of Y^I that contains y . Put $U_y := \mathcal{V}_{y,\eta}^{\text{an}} \cap \pi_{\mathcal{X}}^{-1} Z_y^{I(y)}$. For ω_y , there are two cases:

- (1) If $p(y) = 0$, that is, $Z_y = Y^i$ for some i , then we set

$$\omega_y = \tau \left(\frac{1}{q} \{f_i^\sharp\} \right) |_{U_y}$$

where f_i^\sharp is a lift of $f_i|_{\mathcal{V}_y \cap Y^i}$ to an invertible regular function on $\mathcal{V}_{y,\eta}$.

- (2) If $p(y) > 0$, then we set

$$\omega_y = \tau \left(\frac{1}{q} \prod_{j=1}^{p(y)} g_j^{a(Z_y^{\{i_0, i_j\}})} \right) |_{U_y}$$

where $I(y) = \{i_0 < \dots < i_{p(y)}\}$. Note that $Z_y^{\{i_0, i_j\}} \in \mathcal{D}^1$.

We claim that $\{(U_y, \omega_y)\}$ patch together to give an element in $H^0(X^{\text{an}}, \mathcal{L}_{X^{\text{an}}}^1)$, which we denote by $T_{\mathcal{X}} D$. Given two points y, y' of Y , it amounts to showing that $\omega_y = \omega_{y'}$ on $U_y \cap U_{y'}$. If one of y, y' is in Case (1), then the compatibility can be shown in the same way as in the proof of Lemma 5.7. In particular, ω_y does not depend on the choice of the lift f_i^\sharp in Case (1). If both y and y' are in Case (2), then it is a straightforward consequence of the fact that $\delta_1^* D = 0$ (4.3). We leave those details to the reader.

By construction, $T_{\mathcal{X}}$ is apparently linear and injective. The theorem is proved. \square

6. APPENDIX: STAR-SHAPED INTEGRATION ON SIMPLEX

In this appendix, we prove a technical result used in the proof of Lemma 5.9 as the local computation.

Let $n \geq 1$ be an integer. Let $\Delta \subset \mathbf{R}^{n+1}$ be the standard (closed) simplex of dimension n . If we denote by x_0, \dots, x_n the standard coordinates on \mathbf{R}^{n+1} , then

$$\Delta = \{(x_0, \dots, x_n) \mid x_0 + \dots + x_n = 1, x_i \geq 0\}.$$

For $p \geq 0$, denote by $\Omega^p(\Delta)$ the space of (smooth) p -differential forms on Δ and $\Omega_{\text{cl}}^p(\Delta)$ the subspace of closed forms. Let ω be an element of $\Omega^p(\Delta)$, then for every $0 \leq i \leq n$, ω can be uniquely written as

$$\omega = \sum_{I \subset \{0, \dots, n\} \setminus \{i\}, |I|=p} f_I(x) dx_I$$

where f_I is a continuous function on Δ , smooth in the interior of Δ . We say that ω has *constant coefficients* if f_I is a constant function for every such I . Note that this property is independent of the choice of i . Let $\Omega_{\text{cons}}^p(\Delta) \subset \Omega_{\text{cl}}^p(\Delta)$ be the subspace of forms with constant coefficients.

For every nonempty subset $J \subset \{0, \dots, n\}$, let Δ^J be the face of Δ generated by $\{P^j \mid j \in J\}$, where P^j is the point with coordinates $x_i = \delta_{ij}$, and let P^J be the barycenter of Δ^J . For every point $P \in \Delta$, Δ is star-shaped with respect to P . Thus, one has the *star-shaped integration map*:

$$\mathcal{I}_P: \Omega^p(\Delta) \rightarrow \Omega^{p-1}(\Delta)$$

for $p \geq 1$. We briefly recall the definition. Consider the map $\gamma_P: \Delta \times [0, 1] \rightarrow \Delta$ sending (x, t) to $(1-t)P + tx$. Then

$$\mathcal{I}_P(\alpha) = \int_0^1 \left\langle \frac{\partial}{\partial t}, \gamma_P^* \alpha \right\rangle dt,$$

where $\langle \cdot, \cdot \rangle$ denotes the contraction. If $d\alpha = 0$, then $d(\mathcal{I}_P(\alpha)) = \alpha$.

Let F be an abelian group. For every $r \geq 0$, put

$$C^r(F) = \text{Map}(\{J \subset \{0, \dots, n\} \mid |J| = r+1\}, F).$$

We have a coboundary map $\delta: C^r(F) \rightarrow C^{r+1}(F)$ similar to the one for Čech cocycles. For $p-r \geq 1$, we define the map

$$(6.1) \quad \mathcal{I}: C^r(\Omega^{p-r}(\Delta)) \rightarrow C^r(\Omega^{p-r-1}(\Delta))$$

sending ω_r in the source to $\mathcal{I}\omega_r$ in the target such that $(\mathcal{I}\omega_r)(J) = \mathcal{I}_{P^J}(\omega_r(J))$.

Proposition 6.1. *Let $1 \leq p \leq n$ be an integer. Let β be an element in $\Omega_{\text{cons}}^p(\Delta)$. We define $\beta_0 \in C^0(\Omega^p(\Delta))$ by $\beta_0(\{j\}) = \beta$ for every $0 \leq j \leq n$. For $1 \leq r \leq p$, we inductively define $\beta_r \in C^r(\Omega^{p-r}(\Delta))$ by the formula $\beta_r = \delta(\mathcal{I}\beta_{r-1})$. Then*

- (1) β_r belongs to $C^r(\Omega_{\text{cons}}^{p-r}(\Delta))$ for $0 \leq r \leq p$, and in particular $\beta_p \in C^p(\mathbf{R})$; and
- (2) we have the formula

$$\beta|_{\Delta^I} = (-1)^{p(p+1)/2} p! \cdot \beta_p(I) dx_{i_1} \wedge \dots \wedge dx_{i_p}$$

for every subset $I = \{i_0 < \dots < i_p\} \subset \{1, \dots, n\}$.

Proof. We fix some $i \in \{0, \dots, n\}$ and write

$$(6.2) \quad \beta = \sum_{I \subset \{0, \dots, n\} \setminus \{i\}, |I|=p} f_I dx_I$$

for some $f_I \in \mathbf{R}$. Regard β as a p -form on \mathbf{R}^{n+1} with constant coefficients, which we denote by $\beta' \in \Omega_{\text{cons}}^p(\mathbf{R}^{n+1})$. For a point $Q \in \mathbf{R}^{n+1}$ and $r \geq 1$, denote by

$$\mathcal{I}'_Q: \Omega^r(\mathbf{R}^{n+1}) \rightarrow \Omega^{r-1}(\mathbf{R}^{n+1})$$

the corresponding star-shaped integral on \mathbf{R}^{n+1} with respect to Q . If Q belongs to Δ , then for $\alpha \in \Omega^r(\mathbf{R}^{n+1})$, we have

$$(6.3) \quad \mathcal{I}'_Q(\alpha)|_{\Delta} = \mathcal{I}_Q(\alpha|_{\Delta}).$$

Suppose that $Q = (q_0, \dots, q_n)$; we define a map

$$\mathcal{C}_Q: \Omega_{\text{cons}}^r(\mathbf{R}^{n+1}) \rightarrow \Omega_{\text{cons}}^{r-1}(\mathbf{R}^{n+1})$$

sending α to the contraction of α by $\sum_{j=0}^n q_j \frac{\partial}{\partial x_j}$ (from left). For example,

$$\mathcal{C}_{P^j}(\mathrm{d}x_0 \wedge \dots \wedge \mathrm{d}x_{r-1}) = \begin{cases} (-1)^j \mathrm{d}x_0 \wedge \dots \wedge \widehat{\mathrm{d}x_j} \wedge \dots \wedge \mathrm{d}x_{r-1}, & 0 \leq j \leq r-1; \\ 0, & r \leq j \leq n. \end{cases}$$

Denote by O the origin of \mathbf{R}^{n+1} . It is an elementary exercise to see that for $\alpha \in \Omega_{\text{cons}}^r(\mathbf{R}^{n+1})$, we have

$$(6.4) \quad \mathcal{I}'_Q(\alpha) - \mathcal{I}'_O(\alpha) = -\frac{1}{r} \mathcal{C}_Q(\alpha).$$

For $0 \leq r \leq p$, we define an element $\beta'_r \in C^r(\Omega_{\text{cons}}^{p-r}(\mathbf{R}^{n+1}))$ by the formula

$$(6.5) \quad \beta'_r(I) = \frac{(-1)^r}{p(p-1) \cdots (p-r+1)} \sum_{j=0}^r (-1)^j (\mathcal{C}_{P^{i_0}} \circ \dots \circ \widehat{\mathcal{C}_{P^{i_j}}} \circ \dots \circ \mathcal{C}_{P^{i_r}})(\beta').$$

for $I = \{i_0 < \dots < i_r\}$. We claim that

$$(6.6) \quad \beta'_r = \delta(\mathcal{I}' \beta'_{r-1})$$

where $\mathcal{I}': C^r(\Omega_{\text{cons}}^{p-r}(\mathbf{R}^{n+1})) \rightarrow C^r(\Omega_{\text{cons}}^{p-r-1}(\mathbf{R}^{n+1}))$ is defined similarly as (6.1). In fact, by the definition of δ , we have for $r \geq 1$,

$$(6.7) \quad \delta(\mathcal{I}' \beta'_{r-1}) = \sum_{j=0}^r (-1)^j \mathcal{I}'_{P^{I \setminus \{i_j\}}}(\beta'_{r-1}(I \setminus \{i_j\})).$$

As β'_{r-1} is a closed cocycle, we have $\sum_{j=0}^r (-1)^j \beta'_{r-1}(I \setminus \{i_j\}) = 0$. Thus

$$\begin{aligned} (6.7) &= \sum_{j=0}^r (-1)^j (\mathcal{I}'_{P^{I \setminus \{i_j\}}} - \mathcal{I}'_O)(\beta'_{r-1}(I \setminus \{i_j\})) \\ &= \frac{-1}{p-r+1} \sum_{j=0}^r (-1)^j \mathcal{C}_{P^{I \setminus \{i_j\}}}(\beta'_{r-1}(I \setminus \{i_j\})) \quad \text{by (6.4)} \\ (6.8) \quad &= \frac{(-1)^r}{p(p-1) \cdots (p-r+1)} \sum_{j=0}^r (-1)^j \mathcal{C}_{P^{I \setminus \{i_j\}}} \left(\sum_{k=0}^r \epsilon(j, k) \mathcal{C}(j, k)(\beta') \right). \end{aligned}$$

Here, $\epsilon(j, k)$ is set to be $(-1)^k$ (resp. 0, $(-1)^{k+1}$) if $k < j$ (resp. $k = j$, $k > j$), and $\mathcal{C}(j, k)$ is the map $\mathcal{C}_{P^{i_0}} \circ \dots \circ \mathcal{C}_{P^{i_r}}$ with $\mathcal{C}_{P^{i_j}}$ and $\mathcal{C}_{P^{i_k}}$ removed. Since

$$\mathcal{C}_{P^{I \setminus \{i_j\}}} = \frac{1}{r} (\mathcal{C}_{P^{i_0}} + \dots + \widehat{\mathcal{C}_{P^{i_j}}} + \dots + \mathcal{C}_{P^{i_r}}),$$

it is easy to see that

$$(6.8) = \frac{(-1)^r}{p(p-1) \cdots (p-r+1)} \sum_{j=0}^r (-1)^j \mathcal{C}(j, j)(\beta) = \beta'_r(I).$$

Thus (6.6) holds. By (6.3), we have $\beta_r = \beta'_r|_{\Delta}$ which belongs to $C^r(\Omega_{\text{cons}}^{p-r}(\Delta))$. Thus (1) follows.

Now we compute the value of $\beta_p(I) = \beta'_p(I)$ for $I = \{i_0 < \cdots < i_p\}$. Note that the representative β (6.2) can be chosen with respect to an arbitrary $i \in \{0, \dots, n\}$. In particular, we may take $i = i_0$. By (6.5), we have

$$\beta_p(I) = \frac{(-1)^p}{p!} (\mathcal{C}_{P^{i_1}} \circ \cdots \circ \mathcal{C}_{P^{i_p}})(\beta) = \frac{(-1)^{p(p+1)/2}}{p!} (\mathcal{C}_{P^{i_p}} \circ \cdots \circ \mathcal{C}_{P^{i_1}})(\beta).$$

Thus (2) follows as

$$\beta|_{\Delta^I} = (\mathcal{C}_{P^{i_p}} \circ \cdots \circ \mathcal{C}_{P^{i_1}})(\beta) \, dx_{i_1} \wedge \cdots \wedge dx_{i_p}.$$

□

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